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ON THE DUCTILITY OF LAMINATED MATERIALS

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Abstract—A laminated material is one of the few composite systems for which the effective constitutive behavior can be determined exactly. This is well known for laminated composites with linearly elastic phases in prescribed volume fractions. For these composites, explicit expressions for the effective moduli have been available for at least 30 years. However, it appears that corresponding expressions for the effective energy functions of laminated composites with phases exhibiting nonlinear constitutive behavior are currently unavailable. In this paper, we make straightforward use of a new variational procedure, recently developed by one of the authors, to obtain simple expressions for the effective energy functions of laminated composites with isotropic ductile phases in prescribed volume fractions. The same expressions are given an alternative derivation, starting directly from the classical variational principles. Explicit results are then computed for ductile/brittle systems, such as aluminum/alumina laminates, and also for laminated composites made up of two perfectly plastic phases with different yield stresses. The results—which are representative of other anisotropic geometries, such as fiber-reinforced solids—exhibit a strong coupling between different loading modes that is enhanced by material nonlinearity.

1. INTRODUCTION

This work is concerned with the determination of the effective constitutive behavior of laminated composite materials with plastically deforming phases in prescribed volume fractions. We will deal with the *exact* effective behavior of such materials, and therefore, we will exclude from our consideration the so-called approximate theories of laminated plates (Christensen, 1979). In the context of linear elasticity, laminated composites have been studied extensively in the literature. Postma (1955) and White and Angona (1955) concerned themselves with the study of two-phase, periodic laminates in connection with wave propagation in stratified media. Backus (1962) extended their results to multi-phase, nonperiodic composites, again in the wave propagation context. The extension to anisotropic layers was considered by Walpole (1969) for aligned, transversely isotropic phases, and by Chou *et al.* (1972) and Pagano (1974) for more general anisotropy of the phases. Recently, Norris (1991) has developed alternative expressions for the effective moduli tensor of laminated composites with generally anisotropic phases, exploiting the interior and exterior projection tensors of Hill (1972, 1983). Other related works include iterative formulae developed by Francfort and Murat (1986) in the context of linear elasticity allowing simple expressions for the effective moduli of multi-sandwich structures (laminates embedded within laminates of different orientations). These microstructures are of theoretical value in the demonstration of the optimality of bounds for the effective properties of composite materials with more general microstructures [see Kohn (1987) and Lipton (1991b)]. In spite of the large level of activity for linear laminates, briefly summarized above, the theory of nonlinear laminated composites does not seem to have been developed very much. To the knowledge of the authors, the only work thus far in this direction is a generalization of the Francfort–Murat formula for simple laminated materials with one nonlinear phase and one *linear* phase due to Kohn (1990) and Milton (1990). Nonlinear results do exist, however, within the context of the approximate laminated plate theories.

The justification of the study of nonlinear laminated composites may be partially understood in terms of the following considerations. First, it is a configuration of practical importance: for instance, the use of linear laminated theories in geophysical applications is well known, but it is also known that the properties of the materials composing the surface of the Earth may exhibit nonlinear constitutive behavior, particularly, deep within the surface, where the materials are subject to large compressive stresses. Second, we will

find that the laminated microstructure is illustrative of the significant coupling that may arise in nonlinear, anisotropic materials between different loading modes. This nonlinear coupling is also observed in other more complex microstructures, such as fiber-reinforced composites. Third, the laminated microstructure corresponds to the simplest possible type of anisotropic composite with nonlinear phases, in the sense that *exact* results may be obtained for its effective properties, as will be shown herein. In this connection, the present work should be considered in the light of a research program attempting to characterize the effective properties of nonlinear composites in general. Thus far, methods have been developed for understanding the effective behavior of nonlinear composites by Talbot and Willis (1985), and by Ponte Castañeda (1991a, 1992). These methods, although different in essence, can be shown to yield exactly the same results in some cases, but the new method is more general than the first in that it can be used to obtain estimates *other than* Hashin–Shtrikman bounds and self-consistent estimates [see Willis (1991) and Ponte Castañeda (1992)]. Bounds and estimates for the effective properties of nonlinear composites with isotropic overall symmetries have been obtained by Ponte Castañeda and Willis (1988), and Willis (1989) making use of the Talbot–Willis method, and by Ponte Castañeda (1991a, b) making use of the new method. Results for fiber-reinforced composites have also been developed very recently by Talbot and Willis (1991), and by Ponte Castañeda (1992) and deBotton and Ponte Castañeda (1992).

The rest of the paper is structured as follows. In Section 2 the definition of effective properties is reviewed, and their variational characterization is given in terms of both the classical and new variational principles of Ponte Castañeda (1991a, 1992). In Section 3 general nonlinear laminated composites are considered, and general formulae are derived in Sections 4 and 5 for the effective properties of incompressible and compressible laminates, respectively. Additionally, in Section 6, more specific results are given for two-phase laminates. In particular, the cases of ductile materials reinforced by linearly elastic layers, and of laminates with two perfectly plastic phases are considered. Finally, some additional relevant results are given in four appendices; in particular, in Appendix IV, an alternative derivation is given of the results of Sections 4 and 5 using the classical variational principle.

2. EFFECTIVE PROPERTIES AND THEIR VARIATIONAL CHARACTERIZATION

In this section, we are interested in the characterization of the effective, or overall, constitutive behavior of composites materials with plastically deforming phases. For our purposes, a *composite* is a heterogeneous material with two distinct length scales: one macroscopic, L, describing the gross size of the specimen and the scale of variation of the applied loading conditions, and a microscopic scale, l, characterizing the size of the typical inhomogeneity, such that $l \ll L$. More precise definitions can be found in the review article by Kohn (1987).

For simplicity, the constitutive behavior of the phases will be characterized by the deformation theory of plasticity, or equivalently by nonlinear infinitesimal elasticity. However, with minor changes in notation, the results of the analyses of this paper will also be relevant to the high-temperature creeping behaviour of composite laminates. Additionally, the results can be used in an *approximate* fashion to suggest yield functions for laminated composites in the context of the incremental theory of plasticity, as suggested by Duva and Hutchinson (1984), and other investigators.

In the following description of effective properties, the composite is assumed to occupy a domain of unit volume Ω, with boundary $\partial\Omega$. Then, the nonlinear plastic behavior of the composite is characterized by means of a complementary-energy density function, $U(\mathbf{x}, \boldsymbol{\sigma})$, depending on the position vector $\mathbf{x}$ and the stress field $\boldsymbol{\sigma}(\mathbf{x})$, in such a way that the strain field $\boldsymbol{\varepsilon}(\mathbf{x})$ is given by

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \frac{\partial U(\mathbf{x}, \boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}. \tag{1}$$

Following Hill (1963), we define the *effective* constitutive behavior of the heterogeneous solid in terms of the analogous relation

$$\bar{\boldsymbol{\varepsilon}} = \frac{\partial \tilde{U}}{\partial \bar{\boldsymbol{\sigma}}}, \tag{2}$$

where $\bar{\boldsymbol{\varepsilon}}$ denotes the mean value of the strain field over Ω, and $\tilde{U}$ refers to the normalized (recall that Ω has unit volume) complementary-energy function of the solid when subjected to the *uniform constraint* boundary condition

$$\boldsymbol{\sigma}\mathbf{n} = \bar{\boldsymbol{\sigma}}\mathbf{n}, \quad \mathbf{x} \in \partial\Omega, \tag{3}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, and $\bar{\boldsymbol{\sigma}}$ is a constant, symmetric tensor. We recall that under this type of boundary condition, the mean value of the stress over Ω is precisely $\bar{\boldsymbol{\sigma}}$.

The *effective* complementary energy function of the composite, $\tilde{U}$, can be obtained directly in terms of the principle of minimum complementary energy by means of

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in S(\bar{\boldsymbol{\sigma}})} \int_{\Omega} U(\mathbf{x}, \boldsymbol{\sigma})\, \mathrm{d}v, \tag{4}$$

where

$$S(\bar{\boldsymbol{\sigma}}) = \{\boldsymbol{\sigma} \mid \nabla \cdot \boldsymbol{\sigma} = 0 \text{ in } \Omega, \text{ and } \boldsymbol{\sigma}\mathbf{n} = \bar{\boldsymbol{\sigma}}\mathbf{n} \text{ on } \partial\Omega\} \tag{5}$$

is the set of *statically* admissible stress fields. Note that the first set of conditions in (5) are the equilibrium equations, and that the minimizing conditions (Euler–Lagrange equations) of (4) are the compatibility equations. Further, composite materials typically exhibit sharp interfaces across which the material properties are discontinuous, although the phases are *assumed* to be perfectly bonded. Therefore, across such interfaces, the equilibrium equations must be reinterpreted in terms of continuity of the traction stresses, and correspondingly the compatibility equations must be replaced by continuity of the tangential components of the strain tensor.

We note that, given relation (2) in terms of $\tilde{U}$, the problem of characterizing the effective behavior of the composite reduces to that of determining $\tilde{U}$. However, while in principle $\tilde{U}$ can be computed from (4); in practice, this variational principle is not very useful for two reasons. First, usually the microstructure of a typical composite is not completely specified; and second, the problem described by (4) is a nonlinear one on account of the nonlinear behavior of the constituent phases. For the problem of interest in this paper, the first issue is not a concern because the phase volume fractions suffice to characterize the microstructure of a laminated composite material. However, the second issue presents real difficulties. For this reason, we describe next a new variational principle, introduced recently by Ponte Castañeda (1991a), which deals precisely with the problem of constitutive nonlinearity. This is accomplished by expressing the effective energy function of the *nonlinear* composite in terms of a variational statement involving the effective energy functions of the class of *linear* comparison composites. Thus, the new variational principle allows the extension of well-known results for linear composites to corresponding results for nonlinear ones. In this paper, we will make use of this variational principle, and of well-known results for the effective properties of linearly elastic laminates, to determine the effective constitutive behavior of ductile laminates. Before proceeding with this task, we briefly review the new variational principle.

The new variational principle for the effective energy of the composite $\tilde{U}$ is obtained by means of the Legendre transformation, applied to a modified set of variables. We will assume that the heterogeneous solid is locally *isotropic*, such that

$$U(\mathbf{x}, \boldsymbol{\sigma}) = \psi(\mathbf{x}; \tau_{\mathrm{e}}, \sigma_{\mathrm{m}}), \tag{6}$$

where ψ is a non-negative function, satisfying the condition that $\psi(\mathbf{x}; 0, 0) = 0$ for all $\mathbf{x}$. Additionally, ψ is convex in the variables [*cf.* (A2)]

$$\sigma_m = \tfrac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} \quad \text{and} \quad \tau_e = \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'}, \tag{7}$$

denoting the *mean* and *effective* (in their plasticity usages) stresses, respectively, where $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \sigma_m \mathbf{I}$ is the stress deviator tensor. We note that form (6) is *not* the most general form for the energy function of a nonlinear isotropic solid (we could also have dependence on the determinant of $\boldsymbol{\sigma}$), but this form is still general enough to cover the usual plasticity models of interest here. Further, we will assume that the growth in U as the magnitude of the stress tensor becomes large is stronger than quadratic. This is of course consistent with the ductile behavior of the material.

The new variational principle is obtained then in terms of the following expression for the energy-density function of the heterogeneous solid, namely,

$$U(\mathbf{x}, \boldsymbol{\sigma}) = \max_{\mu_0, \kappa_0 \geq 0} \{U_0(\mathbf{x}, \boldsymbol{\sigma}) - V(\mathbf{x}; \mu_0, \kappa_0)\}, \tag{8}$$

where U_0 is the energy-density function of a linearly elastic comparison solid with shear modulus μ_0, and bulk modulus κ_0, such that

$$U_0(\mathbf{x}, \boldsymbol{\sigma}) = \frac{1}{2\mu_0}\tau_e^2 + \frac{1}{2\kappa_0}\sigma_m^2, \tag{9}$$

and where

$$V(\mathbf{x}; \mu_0, \kappa_0) = \max_{\boldsymbol{\sigma}} \{U_0(\mathbf{x}, \boldsymbol{\sigma}) - U(\mathbf{x}, \boldsymbol{\sigma})\}. \tag{10}$$

(Note that the maximum in the above function is usually bounded, because of the stronger than quadratic assumption on U.) These expressions are obtained by means of the changes of variables, $v_e = \tau_e^2$ and $v_m = \sigma_m^2$, which lead to the definition of a nonnegative function f, such that

$$f(\mathbf{x}; v_e, v_m) = \psi(\mathbf{x}; \tau_e, \sigma_m). \tag{11}$$

Then, expression (10) is nothing more than the Legendre dual of f; in fact, we have that $V(\mathbf{x}; \mu_0, \kappa_0) = f^*(\mathbf{x}; p_e, p_m)$, with $p_e = 1/(2\mu_0)$ and $p_m = 1/(2\kappa_0)$. Here, f^* is the Legendre transform of f, given by

$$f^*(\mathbf{x}; p_e, p_m) = \max_{v_e, v_m \geq 0} \{p_e v_e + p_m v_m - f(\mathbf{x}; v_e, v_m)\}, \tag{12}$$

and (8) is a statement of Legendre duality *for convex* f (i.e. $f^{**} = f$, but written in terms of U and V). For details, we refer the reader to Ponte Castañeda (1992).

The new variational principle is then obtained essentially by inserting expression (8) for U into the principle of minimum complementary energy (4), and interchanging the order of the minimum over the set of admissible stresses with the maximum over the comparison moduli. The result may be expressed in the form (Ponte Castañeda, 1992)

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \max_{\mu_0(\mathbf{x}), \kappa_0(\mathbf{x}) \geq 0} \left\{\tilde{U}_0(\bar{\boldsymbol{\sigma}}) - \int_\Omega V(\mathbf{x}; \mu_0(\mathbf{x}), \kappa_0(\mathbf{x}))\, \mathrm{d}x\right\}, \tag{13}$$

where

$$\tilde{U}_0(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in S(\bar{\boldsymbol{\sigma}})} \int_\Omega U_0(\mathbf{x}, \boldsymbol{\sigma})\, \mathrm{d}x \tag{14}$$

is the effective energy of the linear comparison composite. We emphasize that expression

(13) is a variational principle in its own right since it involves an infinite-dimensional optimization over the set of nonnegative functions $\mu_0(\mathbf{x})$ and $\kappa_0(\mathbf{x})$. Thus, even if we had an explicit expression for the effective energy function of the linear comparison composite $\tilde{U}_0$ [not an easy calculation in general for arbitrary $\mu_0(\mathbf{x})$ and $\kappa_0(\mathbf{x})$], the above variational principle would still be difficult to implement. However, we will see that for a laminated composite, the above problem simplifies dramatically. Similarly, it was shown by Ponte Castañeda (1991a, 1992) that the above variational principle can also be utilized in an approximate fashion to compute bounds for the effective properties of nonlinear composites with more general microstructures. Additionally, in the same references, dual versions of (13) are also given in terms of the minimum potential energy of the composite; however, in this paper we prefer to use the above formulation due to the fact that it is easier to express the stress/strain relation for a ductile material in terms of the complementary energy-density function U than in terms of its Legendre counterpart, the energy-density function $W = U^*$.

3. APPLICATION OF THE NEW VARIATIONAL PRINCIPLE TO A NONLINEAR LAMINATED COMPOSITE

In this section, we specialize the general formulation of the previous section to the case of laminated composites. Such materials consist of n homogeneous, isotropic phases occupying nonintersecting layered regions $\Omega^{(r)}(r = 1, 2, \ldots, n)$, with union Ω and with normal $\mathbf{n}$. The complementary energy-density function for the laminated material is then expressible in the form

$$U(\sigma, \mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x} \cdot \mathbf{n}) U^{(r)}(\sigma), \tag{15}$$

where $\chi^{(r)}(\mathbf{x} \cdot \mathbf{n})$ (equal to 1 for $\mathbf{x}$ in phase r, and 0 otherwise) is the characteristic function of phase r, and $U^{(r)}(\sigma) = \psi^{(r)}(\tau_e, \sigma_m)$ is the corresponding homogeneous, isotropic energy-density function. Also the volume fraction $c^{(r)}$ of each phase is determined by the corresponding characteristic functions $\chi^{(r)}$ via the relation

$$c^{(r)} = \int_{\Omega} \chi^{(r)}(\mathbf{x} \cdot \mathbf{n}) \, \mathrm{d}x. \tag{16}$$

We remark that a laminated composite with *perfectly bonded*, isotropic phases possesses *transversely isotropic* symmetry (with transverse direction $\mathbf{n}$). In some sense, it represents the simplest composite material with transverse isotropy; other examples of practical importance include fiber-reinforced materials with isotropic constituent phases. These will be considered elsewhere. Because of the particular type of anisotropy involved in laminated composites, we have included in Appendix A a brief summary, largely after Walpole (1981), of the appropriate invariants and other useful definitions for transversely isotropic materials.

The computation of the effective energy-density function of a laminated composite is made easy by the following property of laminated composites. If the thickness of the typical layer is small compared to the size of the laminate (i.e. if the laminate—linear or nonlinear—is a *composite* in the sense defined in Section 2), then, away from a boundary-layer region close to the boundary of the composite, the fields are constant within each layer (a different constant in each layer). Therefore, the problem of determining the effective energy function of a laminated composite reduces to that of determining the constant fields within each phase of the composite by imposition of the appropriate jump conditions (continuity of traction stresses and tangential strains) across the interfaces between the different layers, as well as the averaging conditions stated in Section 2. Thus, the problem of determining the effective energy function of a laminated composite, unlike the corresponding problem for a general composite, simplifies to an algebraic one. Although, in principle, the resulting problem can always be solved; in practice, it may be difficult to obtain explicit results

because the jump conditions take the form of complicated sets of nonlinear algebraic equations. However, if the composite is made up of linear phases (with quadratic energy functions in each phase), the jump conditions are also linear and they can be solved in closed form, as discussed in Section 1.

The results for the effective energy functions of linear laminated composites are given in the next two sections; in the balance of this section, we make use of the new variational principle (13) to determine an *exact* expression for the effective energy function of a nonlinear laminated composite $\tilde{U}$, written in terms of the effective energy functions $\tilde{U}_0$ of the class of linearly elastic comparison laminates. This is accomplished by noting that, for a laminate, the minimizing comparison moduli functions $\mu_0(\mathbf{x})$ and $\kappa_0(\mathbf{x})$ in (13) must be constant within each phase. Therefore, it suffices to optimize with respect to the set of *constant* [over each phase $r(r = 1, \ldots, n)$] comparison moduli, $\mu_0^{(r)}$ and $\kappa_0^{(r)}$. Thus, we have that

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \max_{\mu_0^{(r)}, \kappa_0^{(r)} > 0} \left\{ \tilde{U}_0(\bar{\boldsymbol{\sigma}}) - \sum_{s=1}^{n} c^{(s)} V^{(s)}(\mu_0^{(s)}, \kappa_0^{(s)}) \right\}, \tag{17}$$

where, from (10),

$$V^{(r)}(\mu_0^{(r)}, \kappa_0^{(r)}) = \max_{\tau_e, \sigma_m} \left\{ \frac{1}{2\mu_0^{(r)}} \tau_e^2 + \frac{1}{2\kappa_0^{(r)}} \sigma_m^2 - \psi^{(r)}(\tau_e, \sigma_m) \right\}, \tag{18}$$

and where $\tilde{U}_0$ is the effective energy–density function of a linearly elastic laminated material made up of n phases in volume fractions $c^{(r)}$, with shear and bulk moduli, $\mu_0^{(r)}$ and $\kappa_0^{(r)}$, respectively.

On the face of it, expressions (17) with (18) for the effective energy function of a nonlinear laminated composite do not appear to offer much of an analytical advantage over the standard procedure of determining the stress fields within each phase (by solving the appropriate nonlinear jump conditions) and putting them directly in the complementary energy principle (4). This is due to the large number of optimizations involved in expressions (17) and (18) (i.e. a total of $4n$ optimizations for an n-phase laminate). However, we shall see in the next two sections that application of the particular form for the effective energy function of a linearly elastic laminate in (17) leads to a simpler optimization problem for the effective energy function of the nonlinear laminate. Further, we observe that, from a computational point of view, it is generally easier to minimize (or maximize) functions than it is to solve nonlinear sets of equations, and therefore, the methods developed in this paper are computationally superior to the standard procedure of solving systems of nonlinear equations (arising from the jump conditions). In Section 4, we begin by considering the simpler case of a laminated composite with incompressible, isotropic phases, and in Section 5, we tackle the more complicated problem of a general laminated composite with compressible, isotropic phases.

4. THE INCOMPRESSIBLE LAMINATED COMPOSITE

In this section, we deal with the special case of laminated composites with *incompressible*, isotropic phases. In this case, the energy–density functions of each phase take the simpler form $U^{(r)}(\boldsymbol{\sigma}) = \psi^{(r)}(\tau_e)$. Then, relations (17) and (18), expressing the effective energy function $\tilde{U}$ of the nonlinear laminate, reduce to

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \max_{\mu_0^{(r)} > 0} \left\{ \tilde{U}_0(\bar{\boldsymbol{\sigma}}) - \sum_{s=1}^{n} c^{(s)} V^{(s)}(\mu_0^{(s)}) \right\}, \quad \text{and} \quad V^{(r)}(\mu_0^{(r)}) = \max_{\tau_e} \left\{ \frac{1}{2\mu_0^{(r)}} \tau_e^2 - \psi^{(r)}(\tau_e) \right\}, \tag{19, 20}$$

where $\tilde{U}_0$ now refers to the effective energy–density function of a linearly elastic laminated

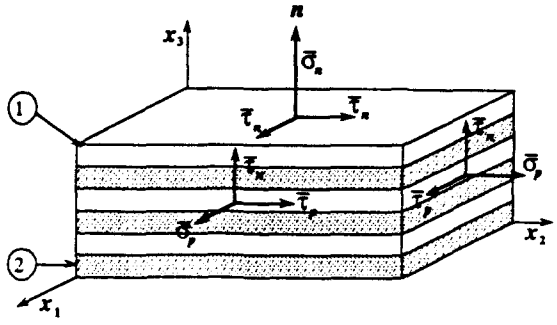

Fig. 1. A two-phase laminated material.

material composed of n incompressible phases with shear moduli $\mu_0^{(r)}$ in prescribed volume fractions $c^{(r)}$.

The effective energy–density function of the linear (incompressible) comparison laminate $\tilde{U}_0$ may be computed from the general results of Walpole (1969), specialized to the case of incompressible, isotropic phases. We obtain

$$\tilde{U}_0(\bar{\sigma}) = \frac{1}{2\bar{\mu}_0}(\bar{\tau}_p^2 + \bar{\tau}_d^2) + \overline{\left(\frac{1}{2\mu_0}\right)}\bar{\tau}_n^2, \tag{21}$$

where the overbars on the moduli denote volume averages (e.g. $\bar{\mu}_0 = \Sigma_{r=1}^n c^{(r)}\mu_0^{(r)}$), and where $\bar{\tau}_n$, $\bar{\tau}_p$ and $\bar{\tau}_d$ are the three transversely isotropic invariants of the applied stress tensor $\bar{\sigma}$ (which is trace-free) corresponding to the three independent modes for an incompressible, transversely isotropic, linear material (see Appendix A and Fig. 1). They are the transverse shear stress [(A5)$_3$], the longitudinal shear stress [(A5)$_4$] and the deviatoric stress [(A5)$_{1,2}$ and (A12)], respectively. We note that the three independent modes for a general incompressible, transversely isotropic material reduce to two independent modes for an incompressible laminated composite (since the transverse and deviatoric modes have the same effective response). Note further that, because of the identity $\bar{\tau}_e^2 = \bar{\tau}_p^2 + \bar{\tau}_d^2 + \bar{\tau}_n^2$ from the section on incompressible materials in Appendix A [(A6)$_2$], we are able to rewrite the first term in brackets in (21) in the form $(\bar{\tau}_e^2 - \bar{\tau}_n^2)$.

With expression (21) for $\tilde{U}_0$, we can now return to the computation of $\tilde{U}$, implied by (19). In this connection, we find that the following identity, proved in Appendix B, is useful in reducing the number of optimizations, namely,

$$\frac{1}{\bar{\mu}_0} = \min_{\omega^{(r)},\bar{\omega}=0}\left\{\sum_{s=1}^n \frac{c^{(s)}}{\mu_0^{(s)}}(1-\omega^{(s)})^2\right\}, \tag{22}$$

where the (constant) optimization variables $\omega^{(r)}(r = 1, \ldots, n)$ are required to satisfy the constraint $\bar{\omega} = 0$. Then, substituting (21), together with (22), into (19) leads to the result

$$\tilde{U}(\bar{\sigma}) = \max_{\mu_0^{(r)}>0}\min_{\omega^{(r)},\bar{\omega}=0}\left\{\sum_{s=1}^n c^{(s)}\left[\frac{1}{2\mu_0^{(s)}}(\tau^{(s)})^2 - V^{(s)}(\mu_0^{(s)})\right]\right\}, \tag{23}$$

where

$$\tau^{(s)} = \sqrt{(\bar{\tau}_p^2 + \bar{\tau}_d^2)(1-\omega^{(s)})^2 + \bar{\tau}_n^2}. \tag{24}$$

We note that, by definition, the functions $-V^{(r)}(\mu_0^{(r)}) = -(f^{(r)})^*(1/(2\mu_0^{(r)}))$ are concave in $1/(\mu_0^{(r)})$, and similarly the variables $\tau^{(r)}$ are convex in $\omega^{(r)}$. Therefore, by the Saddle Point Theorem (Rockafellar, 1970, Corollary 37.3.1), we are allowed to interchange the order of the maximum and the minimum in (23). Further, it follows from (8) [assuming convexity of $f^{(r)}$, where $f^{(r)}(v_e) = \psi^{(r)}(\tau_e)$ with $v_e = \tau_e^2$; see (11)] that

$$\psi^{(r)}(\tau) = \max_{\mu_0^{(r)} > 0} \left\{ \frac{1}{2\mu_0^{(r)}}(\tau)^2 - V^{(r)}(\mu_0^{(r)}) \right\}. \tag{25}$$

Therefore, we conclude from (23) that

$$\tilde{U}(\bar{\sigma}) = \min_{\omega^{(r)}, \bar{\omega}=0} \left\{ \sum_{r=1}^{n} c^{(r)} \psi^{(r)}(\tau^{(r)}) \right\}, \tag{26}$$

where the variables $\tau^{(r)}$ are given by relations (24). Evidently, this form is much simpler than the original form given by (19) and (20); it involves an n-dimensional constrained optimization in place of the $2n$-dimensional optimization problem implied by the original form. However, the linear constraint $\bar{\omega} = 0$ for the n optimization variables $\omega^{(r)}$ $(r = 1, \ldots, n)$ can be embedded into the optimization problem (26) by letting the nth variable $\omega^{(n)}$ be expressed in terms of the other $n-1$ variables $\omega^{(s)}$ $(s = 1, \ldots, n-1)$ via

$$\omega^{(n)} = -\frac{1}{c^{(n)}} \sum_{s=1}^{n-1} c^{(s)} \omega^{(s)}. \tag{27}$$

With this modification, the problem (26) reduces to an $(n-1)$-dimensional optimization problem over the unconstrained variables $\omega^{(s)}$ $(s = 1, \ldots, n-1)$. For instance, for the case of a two-phase laminated composite, the problem (26) reduces to the one-dimensional optimization problem

$$\tilde{U}(\bar{\sigma}) = \min_{\omega} \{ c^{(1)} \psi^{(1)} [\sqrt{(1 - c^{(2)}\omega)^2 (\bar{\tau}_c^2 - \bar{\tau}_n^2) + \bar{\tau}_n^2}] + c^{(2)} \psi^{(2)} [\sqrt{(1 + c^{(1)}\omega)^2 (\bar{\tau}_c^2 - \bar{\tau}_n^2) + \bar{\tau}_n^2}] \}, \tag{28}$$

which is expressed in terms of one (unconstrained) optimization variable ω. Here, we have made the following identifications, $\omega^{(1)} = c^{(2)}\omega$ and $\omega^{(2)} = -c^{(1)}\omega$.

Finally, we remark that simple expressions for the effective stress/strain relations of the nonlinear transversely isotropic laminated composite may be obtained by means of the results of Appendix C. These relations may be written in terms of the incompressible, transversely isotropic invariants of the average strain tensor $\bar{\varepsilon}$, namely, the transverse shear strain $\bar{\gamma}_p$, the deviatoric shear strain $\bar{\gamma}_d$, and the longitudinal shear strain $\bar{\gamma}_n$. These strain invariants are defined in Appendix A, and are completely analogous to the corresponding (incompressible) transversely isotropic invariants of the average stress. Thus, with the help of relations (C7), we may write

$$\begin{aligned}
\bar{\gamma}_p &= \left[\sum_{r=1}^{n} c^{(r)} (1 - \hat{\omega}^{(r)})^2 \frac{1}{\hat{\tau}^{(r)}} \frac{d\psi^{(r)}}{d\tau^{(r)}}(\hat{\tau}^{(r)}) \right] \frac{\bar{\tau}_p}{2}, \\
\bar{\gamma}_d &= \left[\sum_{r=1}^{n} c^{(r)} (1 - \hat{\omega}^{(r)})^2 \frac{1}{\hat{\tau}^{(r)}} \frac{d\psi^{(r)}}{d\tau^{(r)}}(\hat{\tau}^{(r)}) \right] \frac{\bar{\tau}_d}{2}, \\
\bar{\gamma}_n &= \left[\sum_{r=1}^{n} c^{(r)} \frac{1}{\hat{\tau}^{(r)}} \frac{d\psi^{(r)}}{d\tau^{(r)}}(\hat{\tau}^{(r)}) \right] \frac{\bar{\tau}_n}{2},
\end{aligned} \tag{29}$$

where $\hat{\tau}^{(r)} = \tau^{(r)}(\hat{\omega}^{(r)})$, and where the variables $\hat{\omega}^{(r)}$ are the optimized values of the $\omega^{(r)}$ from (26). We note that for the nonlinear laminated composite, there is full coupling between all the distortional (shear) modes. This is different from the situation for the corresponding linear laminated composite [see (21)], where all three modes are uncoupled. As we will see in the ensuing discussions, this inter-mode coupling is one of the intrinsic features of laminated (and other anisotropic) nonlinear composites.

5. THE COMPRESSIBLE LAMINATED COMPOSITE

With the insight gained in the previous section, we attempt in the present section to obtain corresponding results for n-phase laminated composites with nonlinear, isotropic, compressible phases. In this case, we can apply the results (17) and (18) from Section 3 directly; we only require an expression for the effective energy function $\tilde{U}_0$ of the linearly elastic laminate with isotropic, compressible phases in prescribed volume fractions. This energy function may be computed directly from the results of Walpole (1969) for the transversely isotropic moduli of linearly elastic laminated composites. The final result may be written in the form

$$\tilde{U}_0(\bar{\boldsymbol{\sigma}}) = \tilde{U}_1(\bar{\boldsymbol{\sigma}}) + \tilde{U}_2(\bar{\boldsymbol{\sigma}}), \quad \text{where} \quad \tilde{U}_1(\bar{\boldsymbol{\sigma}}) = \frac{1}{2\tilde{\mu}_0}\bar{\tau}_p^2 + \overline{\left(\frac{1}{2\mu_0}\right)}\bar{\tau}_n^2, \tag{30, 31}$$

$$\tilde{U}_2(\bar{\boldsymbol{\sigma}}) = \frac{1}{2\tilde{\eta}_0}\bar{\sigma}_p^2 - \frac{1}{\tilde{\eta}_0}\overline{\left(\frac{3\kappa_0-2\mu_0}{3\kappa_0+4\mu_0}\right)}\bar{\sigma}_p\bar{\sigma}_n + \frac{1}{2}\left[\overline{\left(\frac{3}{3\kappa_0+4\mu_0}\right)} + \frac{1}{\tilde{\eta}_0}\left(\overline{\frac{3\kappa_0-2\mu_0}{3\kappa_0+4\mu_0}}\right)^2\right]\bar{\sigma}_n^2, \tag{32}$$

with $\eta_0 = 9\kappa_0\mu_0/(3\kappa_0+4\mu_0)$, and where $\bar{\sigma}_p$, $\bar{\sigma}_n$, $\bar{\tau}_p$ and $\bar{\tau}_n$ are the four transversely isotropic invariants (up to quadratic in order) of the applied stress $\bar{\boldsymbol{\sigma}}$. They denote, respectively, the in-plane hydrostatic stress, the normal tensile stress, the transverse shear stress and the longitudinal shear stress (see Appendix A and Fig. 1). The reason behind the above splitting of $\tilde{U}_0$ lies in the similarity between the first part of (30), as given by (31), for the distortional (shear) modes of the compressible laminate and relation (21) for the incompressible composite (with $\bar{\tau}_p^2 + \bar{\tau}_d^2$ replaced by $\bar{\tau}_p^2$). Thus, it follows immediately that

$$\tilde{U}_1(\bar{\boldsymbol{\sigma}}) = \min_{\omega_c^{(r)}, \bar{\omega}_c = 0} \left\{ \sum_{s=1}^{n} \frac{c^{(s)}}{2\mu_0^{(s)}} [(1-\omega_c^{(s)})^2\bar{\tau}_p^2 + \bar{\tau}_n^2] \right\}, \tag{33}$$

where the $\omega_c^{(r)}$ are the corresponding optimization variables, and they are subject to the constraint $\bar{\omega}_c = 0$. The second part is more complicated, but it can be shown by straightforward computation that, if $\bar{\sigma}_p \neq 0$, $\tilde{U}_2(\bar{\boldsymbol{\sigma}})$ may be represented in the form

$$\tilde{U}_2(\bar{\boldsymbol{\sigma}}) = \min_{\omega_m^{(r)}, \bar{\omega}_m = 0} \left\{ \sum_{s=1}^{n} \frac{c^{(s)}}{6\mu_0^{(s)}} [\bar{\sigma}_n - (1-\omega_m^{(s)})\bar{\sigma}_p]^2 + \sum_{s=1}^{n} \frac{c^{(s)}}{2\kappa_0^{(s)}} [\tfrac{1}{3}\bar{\sigma}_n + \tfrac{2}{3}(1-\omega_m^{(s)})\bar{\sigma}_p]^2 \right\}, \tag{34}$$

where the optimization variables $\omega_m^{(r)}$ are also subject to the constraint $\bar{\omega}_m = 0$.

By putting together relations (33) and (34), we arrive at the following expression for the linear comparison laminate

$$\tilde{U}_0(\bar{\boldsymbol{\sigma}}) = \min_{\substack{\omega_c^{(r)}, \bar{\omega}_c = 0 \\ \omega_m^{(r)}, \bar{\omega}_m = 0}} \left\{ \sum_{s=1}^{n} \frac{c^{(s)}}{2\mu_0^{(s)}} (\tau_e^{(s)})^2 + \sum_{s=1}^{n} \frac{c^{(s)}}{2\kappa_0^{(s)}} (\sigma_m^{(s)})^2 \right\}, \tag{35}$$

where

$$\tau_e^{(s)} = \sqrt{(1-\omega_c^{(s)})^2\bar{\tau}_p^2 + \bar{\tau}_n^2 + \tfrac{1}{3}[\bar{\sigma}_n - (1-\omega_m^{(s)})\bar{\sigma}_p]^2}, \quad \text{and} \quad \sigma_m^{(s)} = \tfrac{1}{3}\bar{\sigma}_n + \tfrac{2}{3}(1-\omega_m^{(s)})\bar{\sigma}_p. \tag{36}$$

We note that this result is reminiscent of the type of result that one would expect to arise directly from the principle of minimum complementary energy. That this result is indeed directly obtainable from the principle of minimum complementary energy is demonstrated in Appendix D.

Then, following a procedure similar to the one followed in the development of expression (26) for the effective energy function of the nonlinear, incompressible laminated composite, but making use of (17) and (18), we arrive at the following expression for the effective energy function of the nonlinear laminated composite

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\substack{\omega_c^{(r)},\omega_m^{(r)} \\ \bar{\omega}_c=\bar{\omega}_m=0}} \left\{ \sum_{r=1}^{n} c^{(r)} \psi^{(r)}(\tau_e^{(r)}, \sigma_m^{(r)}) \right\}, \tag{37}$$

where $\tau_e^{(r)}$ and $\sigma_m^{(r)}$ are given by (36). Here, we have made use of the Saddle Point theorem allowing the interchange in the order of the minimum over the $\omega_e^{(r)}, \omega_m^{(r)}$ variables with the maximum over the comparison moduli $\mu_0^{(r)}$ and $\kappa_0^{(r)}$. We note that form (37) for the effective energy function $\tilde{U}$ of the nonlinear laminated composite is a direct generalization of form (35) for the effective energy function $\tilde{U}_0$ of the linear comparison laminate. We also note that while the distortional and dilatational modes are not coupled in the linear laminated composite [although the dilational modes are coupled among themselves, as (32) shows], all four modes are strongly coupled for the nonlinear laminated composite.

Having obtained the simple form (37) for the effective energy function of a nonlinear laminate by means of the new variational principle, it can be demonstrated that the same result may be obtained directly from the principle of minimum complementary energy. This alternative derivation of (37) is given in Appendix D. We note in this connection that while the derivation of Appendix D may be physically more appealing than the above derivation, in this paper we have chosen to emphasize the derivation based on the new variational principles for the following reasons. The derivation based on the principle of minimum complementary energy depends on the fact that the fields are constant within different phases in the laminated composite; however, for a more general microstructure, such as a fiber-reinforced composite, the fields are no longer constant within the phases, and the minimum complementary energy approach would not work. On the other hand, deBotton and Ponte Castañeda (1992) have made use of the new variational principles to obtain expressions analogous to expressions (36) and (37) for the effective energy functions of nonlinear fiber-reinforced composites. Thus, the approach based on the new variational principles is more general and that is the reason for emphasizing the new approach, even in the simple case of a laminated composite, where the new approach is not strictly required. In Appendix D, we also show that an alternative form of (37) is possible, which is not subject to the $\bar{\sigma}_p \neq 0$ restriction, although we note that the above form is still valid in the limit as $\bar{\sigma}_p \to 0$ (it is just not valid in a pointwise sense at $\bar{\sigma}_p = 0$, because the optimizing variables $\omega_m^{(r)}$ become unbounded in that limit).

The new representation for the effective energy function of a nonlinear laminated composite $\tilde{U}$ can be seen to involve only a $2n$-dimensional optimization problem with two linear constraints. This is major reduction in order compared with the original expressions (17) and (18) involving a $4n$-dimensional optimization problem. However, as noted in the previous section, further reductions are possible [to a $2(n-1)$-dimensional optimization problem] by embedding the linear constraints directly into the optimization problem (37). For example, for the case of a two-phase composite, we obtain a result involving only a two-dimensional optimization problem prescribed in terms of the variables ω_c, ω_m via

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\omega_c,\omega_m} \{ c^{(1)} \psi^{(1)}(\tau_e^{(1)}, \sigma_m^{(1)}) + c^{(2)} \psi^{(2)}(\tau_e^{(2)}, \sigma_m^{(2)}) \}, \tag{38}$$

where $\tau_e^{(1)}$, $\sigma_m^{(1)}$ and $\tau_e^{(2)}$, $\sigma_m^{(2)}$ are given by relations (36) with $\omega_e^{(1)} = c^{(2)}\omega_c$, $\omega_e^{(2)} = -c^{(1)}\omega_c$, $\omega_m^{(1)} = c^{(2)}\omega_m$ and $\omega_m^{(2)} = -c^{(1)}\omega_m$.

Finally, we remark that simple expressions for the effective stress/strain relations of the transversely isotropic laminated composite may be obtained by means of the results of Appendix C. These may be written in terms of the transversely isotropic invariants of the average strain tensor $\bar{\boldsymbol{\varepsilon}}$, the in-plane hydrostatic strain $\bar{\varepsilon}_p$, the normal tensile strain $\bar{\varepsilon}_n$, the transverse shear strain $\bar{\gamma}_p$, and the longitudinal shear strain $\bar{\gamma}_n$. These are defined in Appendix A, and are completely analogous to the corresponding transversely isotropic invariants of the average stress. Thus, with the help of relations (C8), we may write

$$\bar{\varepsilon}_p = \frac{1}{6} \sum_{r=1}^{n} c^{(r)}(1-\hat{\omega}_m^{(r)}) \left[2 \frac{\partial \psi^{(r)}}{\partial \sigma_m^{(r)}}(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)}) + [(1-\hat{\omega}_m^{(r)})\bar{\sigma}_p - \bar{\sigma}_n] \frac{1}{\hat{\tau}_e^{(r)}} \frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)}) \right],$$

$$\bar{\varepsilon}_n = \frac{1}{3} \sum_{r=1}^{n} c^{(r)} \left[\frac{\partial \psi^{(r)}}{\partial \sigma_m^{(r)}}(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)}) - [(1-\hat{\omega}_m^{(r)})\bar{\sigma}_p - \bar{\sigma}_n] \frac{1}{\hat{\tau}_e^{(r)}} \frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)}) \right],$$

$$\bar{\gamma}_p = \left[\sum_{r=1}^{n} c^{(r)}(1-\hat{\omega}_e^{(r)})^2 \frac{1}{\hat{\tau}_e^{(r)}} \frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)})\right]\frac{\bar{\tau}_p}{2},$$

$$\bar{\gamma}_n = \left[\sum_{r=1}^{n} c^{(r)} \frac{1}{\hat{\tau}_e^{(r)}} \frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)})\right]\frac{\bar{\tau}_n}{2}, \tag{39}$$

where $\hat{\tau}_e^{(r)} = \tau_e^{(r)}(\hat{\omega}_e^{(r)}, \hat{\omega}_m^{(r)})$, $\hat{\sigma}_m^{(r)} = \sigma_m^{(r)}(\hat{\omega}_m^{(r)})$, and where $\hat{\omega}_e^{(r)}, \hat{\omega}_m^{(r)}$ are the optimized values of $\omega_e^{(r)}, \omega_m^{(r)}$ from (37). We note that in this form, the coupling between the distortional and dilatational modes in the nonlinear material become evident since $\bar{\gamma}_p$, $\bar{\gamma}_n$ depend on $\bar{\sigma}_n$, $\bar{\sigma}_p$, and, conversely, $\bar{\varepsilon}_n$, $\bar{\varepsilon}_p$ depend on $\bar{\tau}_n$, $\bar{\tau}_p$.

6. APPLICATION TO LAMINATED COMPOSITES WITH POWER-LAW CONSTITUTIVE BEHAVIOR

In this section, we specialize the results of Sections 4 and 5 for three classes of laminated composites. The first subsection deals with the case of an incompressible laminated material made up of layers of a phase with "linear plus power-hardening" constitutive behavior, reinforced with stiffer layers of a linear-elastic material. In the study of these incompressible laminates, we will emphasize the coupling between different distortional loading modes arising as a consequence of nonlinearity and anisotropy in the laminates. The second subsection is dedicated to the study of a compressible, aluminum/alumina laminate, and the understanding of the dilatational modes is emphasized in this case. The third subsection deals with an incompressible laminated composite made up of two rigid/perfectly plastic phases with different yield stresses; it is interesting to note that, in this special case, completely explicit results are obtained for the effective yield function of the laminate.

6.1. *Incompressible laminated composites*

In this subsection, we consider an incompressible, two-phase laminated composite characterized by the following constitutive laws for the two isotropic phases. Phase 1 is governed by "linear plus power-hardening" constitutive behavior described by the energy-density function

$$\psi^{(1)}(\tau_e) = \int_0^{\sqrt{3}\tau_e} F^{(1)}(s)\,ds, \quad \text{where} \quad F^{(1)}(s) = \varepsilon_0\left\{\frac{s}{\sigma_0} + \left[\left(\frac{s}{\sigma_0}\right)^n - \left(\frac{\sigma_y}{\sigma_0}\right)^n\right]H(s-\sigma_y)\right\}. \tag{40,41}$$

Here H is the unit step function (equal to 0 when $s \leqslant \sigma_y$ and to 1 otherwise), and ε_0, σ_0 are strain, stress normalization factors such that $\sigma_0/\varepsilon_0 = 3\mu^{(1)}$, with $\mu^{(1)}$ denoting the shear modulus of phase 1. Then, the function $F^{(1)}$ represents the uniaxial stress/strain relation of phase 1 under simple tension loading conditions. Thus, the behavior of phase 1 is linear when the uniaxial stress is lower than the yield stress, σ_y, and is linear plus power-hardening for stresses larger than σ_y. The factor $\sqrt{3}$ in (40) is needed in order to fit the isotropic stress invariant τ_e to the uniaxial case. Phase 2 is linear and governed by the quadratic energy-density function

$$\psi^{(2)}(\tau_e) = \frac{1}{2\mu^{(2)}}\tau_e^2, \tag{42}$$

where $\mu^{(2)}$ is the shear modulus of the phase.

With the above constitutive behavior for the two phases (1 and 2), which are prescribed in volume fractions $(1-c^{(2)})$ and $c^{(2)}$, respectively, the effective energy-density function of the incompressible laminated composite may be expressed in dimensionless form via the relation

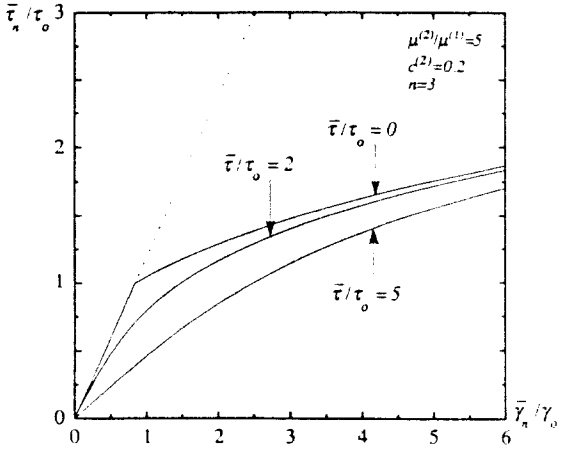

Fig. 2. The relations between the longitudinal shear stress $\bar{\tau}_n$ and strain $\bar{\gamma}_n$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for three different values of the other stress mode $\bar{\tau}$: $\bar{\tau}/\tau_0 = 0$, $\bar{\tau}/\tau_0 = 2$ and $\bar{\tau}/\tau_0 = 5$.

$$\frac{\tilde{U}(\bar{\boldsymbol{\sigma}})}{\tau_0\gamma_0} = G\left\{\frac{\bar{\tau}_p}{\tau_0}, \frac{\bar{\tau}_d}{\tau_0}, \frac{\bar{\tau}_n}{\tau_0}; \frac{\sigma_y}{\sigma_0}, \frac{\mu^{(2)}}{\mu^{(1)}}, n, c^{(2)}\right\}. \tag{43}$$

where the specific form of the function G is determined from (28), and $\tau_0 = \sqrt{\frac{1}{3}}\sigma_0$, $\gamma_0 = \sqrt{3}\varepsilon_0$, so that $\tau_0\gamma_0 = 2\mu^{(1)}$. Then, the relations between the three (incompressible) transversely isotropic stress invariants and the corresponding strain invariants may be computed from (29). These relations are presented in Figs 2–5 for the following values of the four parameters appearing in (43):

$$\frac{\sigma_y}{\sigma_0} = 1, \quad \frac{\mu^{(2)}}{\mu^{(1)}} = 5, \quad n = 3 \quad \text{and} \quad c^{(2)} = 0.2.$$

We recall that there are only two independent modes for the incompressible laminated composites; they are the longitudinal shear stress $\bar{\tau}_n$ and the following combination of the other two shear modes $\sqrt{\bar{\tau}_p^2 + \bar{\tau}_d^2}$ (i.e. the transverse and deviatoric shear modes, respectively). For simplicity, we will refer to this combination of the two modes at $\bar{\tau}$ and to the corresponding combination of the strain modes, $\sqrt{\bar{\gamma}_p^2 + \bar{\gamma}_d^2}$, as $\bar{\gamma}$. Thus, it suffices to consider the relations among the stress modes $\bar{\tau}_n$, $\bar{\tau}$ and the strain modes $\bar{\gamma}_n$, $\bar{\gamma}$ in order to have a complete description of the constitutive behavior of the incompressible laminate. In order to highlight the effect of nonlinearity, results are included in Figs 2–5 in the form of short-

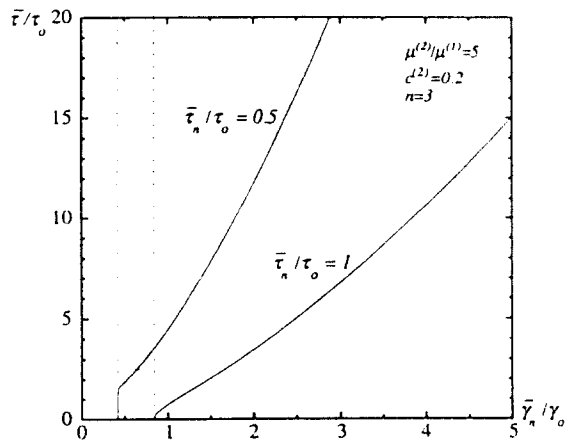

Fig. 3. The inter-relations between the shear stress $\bar{\tau}$ and the longitudinal shear strain $\bar{\gamma}_n$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for two different values of the longitudinal shear stress $\bar{\tau}_n$: $\bar{\tau}_n/\tau_0 = 0.5$ and $\bar{\tau}_n/\tau_0 = 1$.

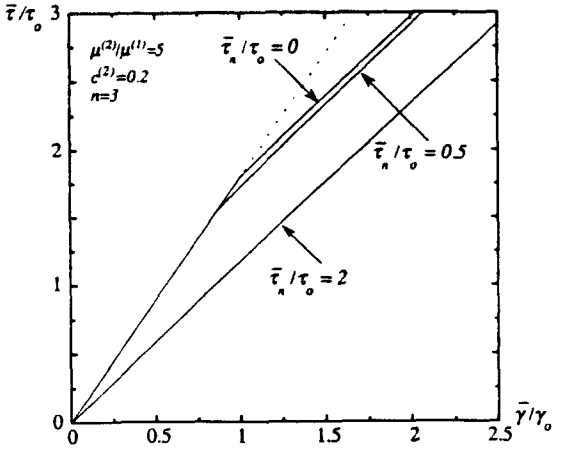

Fig. 4. The relations between the shear stress $\bar{\tau}$ and the corresponding shear strain $\bar{\gamma}$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for three different values of the longitudinal shear stress $\bar{\tau}_n$: $\bar{\tau}_n/\tau_0 = 0$, $\bar{\tau}_n/\tau_0 = 0.5$ and $\bar{\tau}_n/\tau_0 = 2$.

dashed curves for a linear laminated composite with the same shear moduli as the nonlinear laminate. Thus, the phases of this linear *reference* laminate are similar to those of the nonlinear one with the only difference that in phase 1 $\sigma_y = \infty$.

Figure 2 shows a plot of the longitudinal shear stress $\bar{\tau}_n$ versus the longitudinal shear strain $\bar{\gamma}_n$ for three different values of $\bar{\tau}$ ($\bar{\tau}/\tau_0 = 0, 2, 5$). We observe that when there is no preloading of the laminate ($\bar{\tau}/\tau_0 = 0$), the behavior of the stress/strain curve of the nonlinear laminate is initially the same as that of the reference linear laminate (short-dash line) until phase 1 yields. After yielding, the two curves diverge with the nonlinear phase controlling the behavior for large longitudinal shear stresses. That this should be so is seen from the fact that shear parallel to the layers should be controlled by the less stiff phase (in this case, the nonlinear phase). The effect of increasing $\bar{\tau}$ is to saturate the linear range of phase 1, forcing the effective stress/strain curve of the laminate to be controlled by the nonlinear phase even for small values of the longitudinal shear stresses $\bar{\tau}_n$.

Figure 3 shows a plot of $\bar{\tau}$ versus the longitudinal shear strain $\bar{\gamma}_n$ for two different values of the longitudinal shear stress ($\bar{\tau}_n/\tau_0 = 0.5, 1$), and serves to emphasize the coupling between the two shear modes. Thus, a small preload in the form of a longitudinal shear stress $\bar{\tau}_n$ applied to the nonlinear laminate can lead to large increases in the longitudinal strain $\bar{\gamma}_n$ as the other shear stress mode $\bar{\tau}$ is increased; in fact, the growth is unbounded and can be shown to be proportional to $(\bar{\tau}/\tau_0)^{(n-1)/n}$.

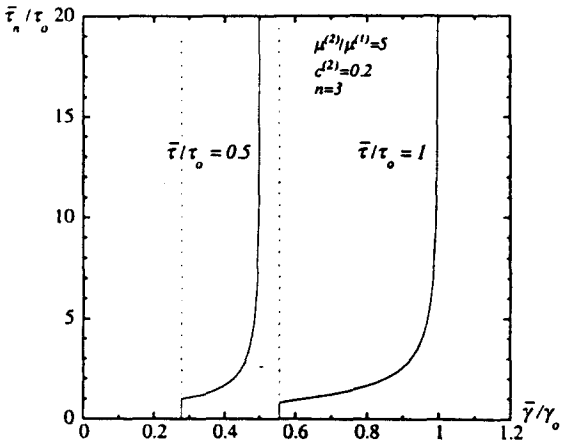

Fig. 5. The inter-relations between the longitudinal shear stress $\bar{\tau}_n$ and the shear strain $\bar{\gamma}$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for two different values of the stress mode $\bar{\tau}$: $\bar{\tau}/\tau_0 = 0.5$, and $\bar{\tau}/\tau_0 = 1$.

Figure 4 shows a plot of the shear stress $\bar{\tau}$ versus the shear strain $\bar{\gamma}$ for three different values of longitudinal shear stress $\bar{\tau}_n$ ($\bar{\tau}_n/\tau_0 = 0, 0.5, 1$). We observe that when there is no preloading of the laminate ($\bar{\tau}_n/\tau_0 = 0$), the behavior of the effective stress strain curve of the nonlinear laminate is initially the same as that of the reference linear laminate (short-dash line) until phase 1 yields. After yielding, however, the two curves diverge with the linear phase controlling the behavior for large shear stresses $\bar{\tau}$. In fact, it can be demonstrated that the slope of the stress strain curve in question reaches an asymptotic value of $2c^{(2)}\mu^{(2)}$ (corresponding to a linear Voigt estimate with $\mu^{(1)} \to 0$) as the shear stress $\bar{\tau}$ becomes large. Evidently, the weaker nonlinear phase is acting as if it was not present for large enough $\bar{\tau}$. The effect of increasing $\bar{\tau}_n$ is then to saturate the linear range of phase 1, reducing the effect of the nonlinear phase on the effective stress strain curve of the composite (the laminate behaves almost linearly with modulus $c^{(2)}\mu^{(2)}$ for sufficiently large preload $\bar{\tau}_n$). That the nonlinear laminate should be controlled by the stiffer linear phase for large magnitudes of the transverse shear stress $\bar{\tau}_p$ (and fixed longitudinal shear stress $\bar{\tau}_n$) is easy to visualize, but that exactly the same behavior should be observed for the deviatoric mode $\bar{\tau}_d$ (the other component of $\bar{\tau}$) is perhaps less intuitive. The reason, however, is related to the Poisson effect. Thus, for example, if the laminate is compressed along the normal direction (which may seem to be controlled by the less stiff nonlinear phase), tensile strains are set up in the plane of the layers, which must be continuous across the phases, thus providing the required stiffening effect in the normal direction (because the linear phase controls the in-plane behavior of the laminate).

Figure 5 shows the relation between the longitudinal shear stress mode $\bar{\tau}_n$ and the strain mode $\bar{\gamma}$ for different values of the shear stress $\bar{\tau}$ ($\bar{\tau}/\tau_0 = 0.5, 2$). We observe that while there is significant coupling between the two modes (by comparison with the linear reference laminate), the coupling is not as significant as in Fig. 3. Thus, the shear strain $\bar{\gamma}$ reaches a maximum level for a given shear preload $\bar{\tau}$ as the longitudinal shear $\bar{\tau}_n$ is increased. This is because the nonlinear phase is dominated by the linear phase in this mode of deformation as observed previously in connection with Fig. 4. The effect of increasing preload $\bar{\tau}$ is to increase (in both absolute and relative terms) the increments in the shear strain $\bar{\gamma}$ with increasing shear stress $\bar{\tau}_n$.

6.2. *The aluminum/alumina laminated composite*

In this subsection, we demonstrate the behavior of a nonlinear, compressible laminated composite made up of aluminum layers reinforced with layers of alumina. Aluminum is a ductile material with uniaxial stress/strain curves that can be approximated by a "linear-plus-power" law with hardening exponent n varying between 4.2 and 5.8. Thus, we will assume the following form for the energy–density function of the aluminum layers (phase 1)

$$\psi^{(1)}(\tau_e, \sigma_m) = \int_0^{\sqrt{3}\tau_e} F^{(1)}(s)\, ds + \frac{1}{2\kappa^{(1)}}\sigma_m^2, \tag{44}$$

where $F^{(1)}$ is the same as in (41), and thus the only difference between (40) and (44) is the compressibility of aluminum accounted for in (44) through the bulk modulus $\kappa^{(1)}$. Alumina (phase 2) is a brittle material that behaves in a linear fashion up to the point of failure. Its energy–density function is represented by

$$\psi^{(2)}(\tau_e, \sigma_m) = \frac{1}{2\mu^{(2)}}\tau_e^2 + \frac{1}{2\kappa^{(2)}}\sigma_m^2, \tag{45}$$

where $\mu^{(2)}$ and $\kappa^{(2)}$ denote the shear and bulk moduli of the alumina, respectively.

With this choice of $\psi^{(1)}$ and $\psi^{(2)}$ (for the behaviors of the two phases), the effective energy–density function of the composite can be represented in dimensionless form via

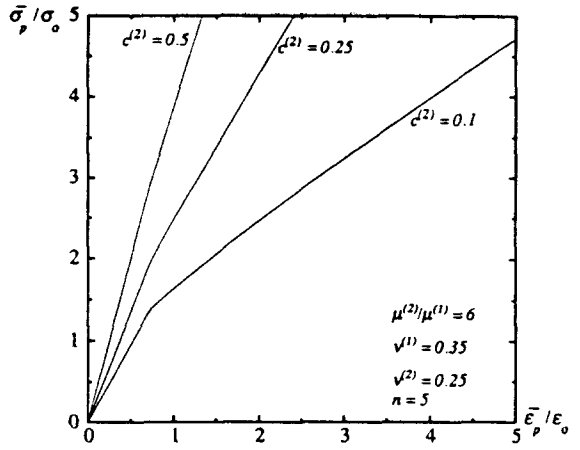

Fig. 6. The relations between the in-plane hydrostatic stress $\bar{\sigma}_p$ and strain $\bar{\varepsilon}_p$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

$$\frac{\tilde{U}(\bar{\sigma})}{\tau_0\gamma_0} = G\left\{\frac{\bar{\sigma}_p}{\sigma_0}, \frac{\bar{\sigma}_n}{\sigma_0}, \frac{\bar{\tau}_p}{\tau_0}, \frac{\bar{\tau}_n}{\tau_0}; \frac{\sigma_y}{\sigma_0}, \frac{\mu^{(2)}}{\mu^{(1)}}, \nu^{(1)}, \nu^{(2)}, n, c^{(2)}\right\}, \tag{46}$$

where G is obtained from (38), $\tau_0 = \sqrt{\frac{1}{3}}\sigma_0$, $\gamma_0 = \sqrt{\frac{3}{4}}\varepsilon_0$, such that $\tau_0/\gamma_0 = 2\mu^{(1)}$, and $\nu^{(1)}$, $\nu^{(2)}$ are the (dimensionless) Poisson's ratios of the two phases defined by

$$\nu^{(r)} = \frac{3\kappa^{(r)} - 2\mu^{(r)}}{6\kappa^{(r)} + 2\mu^{(r)}}.$$

In the results to follow, we will make the following choices (which are representative of the aluminum/alumina composite) for the material parameters in (46):

$$\frac{\sigma_y}{\sigma_0} = 1, \quad \frac{\mu^{(2)}}{\mu^{(1)}} = 6, \quad \nu^{(1)} = 0.35, \quad \nu^{(2)} = 0.25 \text{ and } n = 5.$$

The results are presented in Figs 6–9 in terms of plots of the four transversely isotropic stress modes versus the corresponding strain modes for three different values of the volume fraction of alumina $c^{(2)}$ (0.1, 0.25 and 0.5).

Figure 6 shows a plot of the in-plane hydrostatic stress $\bar{\sigma}_p$ versus the corresponding hydrostatic strain $\bar{\varepsilon}_p$, when all other stress modes vanish, for the three values of the volume

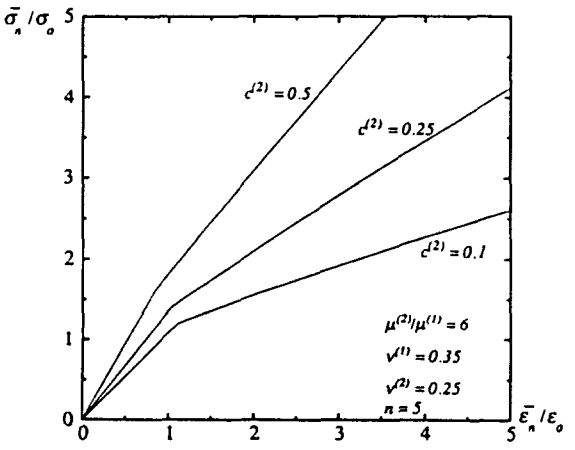

Fig. 7. The relations between the normal tensile stress $\bar{\sigma}_n$ and strain $\bar{\varepsilon}_n$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

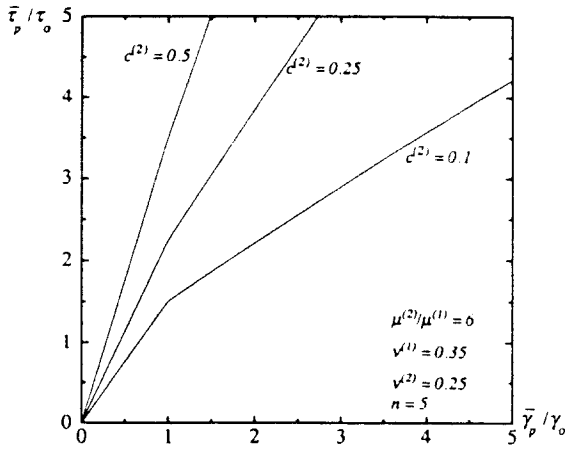

Fig. 8. The relations between the transverse shear stress $\bar{\tau}_p$ and strain $\bar{\gamma}_p$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

fraction of alumina $c^{(2)}$. We observe that the laminate has a linear range with effective modulus $2\bar{\eta}$ (recall that $\eta = 9\kappa\mu/(3\kappa+4\mu)$) up to yielding of the aluminum phase. However, the laminate behaves almost linearly even after yielding with modulus approaching $2c^{(2)}\eta^{(2)}$ for large values of $\bar{\sigma}_p$. This behavior is expected on physical grounds due to the fact that the stiffer material (alumina) should dominate the behavior in tension (compression) parallel to the layers. The effect of increasing volume fractions of alumina is of course to stiffen the effective behavior of the composite.

Figure 7 shows a plot of the normal tensile stress $\bar{\sigma}_n$ versus the corresponding tensile strain $\bar{\varepsilon}_n$, when all other stress modes vanish, for the three previous values of the volume fraction of alumina $c^{(2)}$. The structure of the plots is very similar to that of Fig. 6; however, the effective moduli are different. Before phase 1 reaches yielding, the laminate has uniaxial modulus given by the expression

$$\left[\left(\frac{3}{3\kappa+4\mu}\right)+\frac{1}{\bar{\eta}}\left(\frac{3\kappa-2\mu}{3\kappa+4\mu}\right)^2\right]^{-1},$$

while after the yielding of phase 1, the modulus for large stress $\bar{\sigma}_n$ is reduced to the level

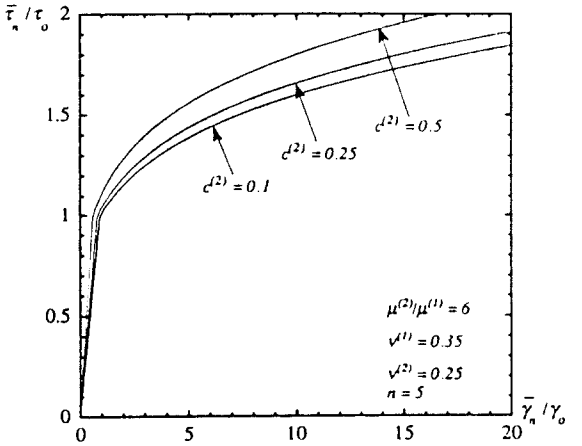

Fig. 9. The relations between the longitudinal shear stress $\bar{\tau}_n$ and strain $\bar{\gamma}_n$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

$$\left[\frac{1}{c^{(2)}\eta^{(2)}}+\overline{\left(\frac{1}{\kappa}\right)}-\frac{4}{3\kappa^{(2)}}\right]^{-1}.$$

In this case, it is not evident that the linear phase should govern the effective behavior of the laminate for large stresses. The reason, however, is the same as discussed in the previous subsection in connection with Fig. 4: continuity of the tangential strains across the interfaces together with the Poisson effect.

Figure 8 shows the corresponding plots for the transverse shear stress $\bar{\tau}_p$ versus the transverse shear strain $\bar{\gamma}_p$, when no other stress modes are present, for the three values of the volume fraction of alumina. In this case, the results are similar to those of Fig. 6 for clear physical reasons: the stiffer phase controls the behavior of the laminate under transverse shear loading.

Figure 9 shows plots of the longitudinal shear stress $\bar{\tau}_n$ versus the corresponding strain mode $\bar{\gamma}_n$, with no other stress modes present. The behavior in this case is dramatically different, as the study of the corresponding case for the incompressible laminate demonstrated earlier (Fig. 2). Thus, after an initial linear range before yielding of phase 1, the weaker nonlinear phase governs the effective behavior of the laminate. In contrast to the other three modes, we observe that the dependence on the volume fraction of alumina is fairly weak, so that the three curves (corresponding to different values of $c^{(2)}$) are quite close to each other.

Clearly, a study of the inter-relations between the different modes would be required to have a complete picture of the effective behavior of the nonliear compressible laminate. However, the behavior of these inter-modal relations is similar to those already explored for the incompressible laminate. Thus, the inter-modes relations that involve the longitudinal shear strain $\bar{\gamma}_n$ are of the form of the relations presented in Fig. 3 while all other inter-modal stress/strain relations are in the form of Fig. 5.

6.3. *The rigid/perfectly plastic laminated composite*

In this subsection, we consider the case of an incompressible laminated composite made up of two rigid/perfectly plastic phases with yield stresses $\tau_0^{(1)}$ and $\tau_0^{(2)}$, chosen such that $\tau_0^{(1)} < \tau_0^{(2)}$, in given volume fractions $c^{(1)}$ and $c^{(2)}$. The behavior of the phases may then be characterized in terms of the convex energy-density functions

$$\psi^{(r)}(\tau_e)=\begin{cases}0, & \tau_e \leqslant \tau_0^{(r)},\\ \infty, & \tau_e > \tau_0^{(r)},\end{cases} \tag{47}$$

($r = 1, 2$), where τ_e denotes the effective shear stress. These energy functions may be obtained directly from pure power-law energy functions of the form

$$\psi^{(r)}(\tau_e)=\frac{1}{n+1}\varepsilon_0\sigma_0^{(r)}\left(\frac{\tau_e}{\tau_0^{(r)}}\right)^{n+1}, \tag{48}$$

in the limit as $n \to \infty$. Further, these energy functions define "yield functions" for the phase materials that may be described in the usual way via

$$\phi^{(r)}(\sigma)=\tau_e-\tau_0^{(r)}=0. \tag{49}$$

Here, we will proceed *formally* and make use of expression (28) to determine an expression for the effective energy function of the laminated material $\tilde{U}$, from which we will be able to determine a yield function for the laminated composite Φ. For a rigorous treatment of homogenization theory for rigid/perfectly plastic composites, and in particular for a discussion concerning the validity of the normality condition for the effective yield

function of the composite, the reader is referred to Suquet (1985). Because of incompressibility and transverse isotropy, we will find that such a yield function may be represented as a curve in the $(\bar{\tau}_n, \bar{\tau})$-space, where $\bar{\tau} = \sqrt{\bar{\tau}_p^2 + \bar{\tau}_q^2}$. Thus, application of (28) leads to the expression

$$\tilde{U}(\bar{\sigma}) = \min_{\omega} \{G(\bar{\tau}_n, \bar{\tau}, \omega)\}, \quad \text{where} \quad G = \begin{cases} 0, & \tau^{(1)} \leqslant \tau_0^{(1)} \text{ and } \tau^{(2)} \leqslant \tau_0^{(2)}, \\ \infty, & \tau^{(1)} > \tau_0^{(1)} \text{ or } \tau^{(2)} > \tau_0^{(2)}, \end{cases} \tag{50, 51}$$

and where $\tau^{(1)} = \sqrt{(1 - c^{(2)}\omega)^2\bar{\tau}^2 + \bar{\tau}_n^2}$ and $\tau^{(2)} = \sqrt{(1 + c^{(1)}\omega)^2\bar{\tau}^2 + \bar{\tau}_n^2}$.

The above optimization problem for ω then reduces to determining all possible combinations of $\bar{\tau}$ and $\bar{\tau}_n$ for which $\tilde{U} = 0$, which in turn defines the yield function for the composite Φ. First, we note that, independent of $\bar{\tau}$ and ω, $\tilde{U}$ can only vanish if

$$\bar{\tau}_n - \tau_0^{(1)} \leqslant 0, \tag{52}$$

for otherwise $\tau^{(1)} \geqslant \bar{\tau}_n > \tau_0^{(1)}$. Thus, inequality (52) is a *necessary* condition for $\tilde{U}$ to vanish. However, the condition (52) is not *sufficient* to ensure that $\tilde{U}$ vanishes since the condition $\tau^{(2)} \leqslant \tau_0^{(2)}$ may be violated. Thus, assuming that condition (52) is satisfied, we ask the question of whether there are values of ω, depending on $\bar{\tau}$ and $\bar{\tau}_n$, such that conditions $\tau^{(1)} \leqslant \tau_0^{(1)}$ and $\tau^{(2)} \leqslant \tau_0^{(2)}$ are satisfied simultaneously. The answer is affirmative, provided that $\bar{\tau}$ and $\bar{\tau}_n$ (for given volume fractions $c^{(1)}$ and $c^{(2)}$) satisfy the condition

$$\bar{\tau} \leqslant c^{(1)}\sqrt{(\tau_0^{(1)})^2 - \bar{\tau}_n^2} + c^{(2)}\sqrt{(\tau_0^{(2)})^2 - \bar{\tau}_n^2}. \tag{53}$$

Thus, conditions (52) and (53) define an effective yield function for the composite, $\Phi = 0$, such that

$$\Phi(\bar{\sigma}) \equiv \begin{cases} \bar{\tau} - [c^{(1)}\sqrt{(\tau_0^{(1)})^2 - \bar{\tau}_n^2} + c^{(2)}\sqrt{(\tau_0^{(2)})^2 - \bar{\tau}_n^2}], & \bar{\tau}_n < \tau_0^{(1)}, \\ \bar{\tau}_n - \tau_0^{(1)}, & \bar{\tau}_n = \tau_0^{(1)}. \end{cases} \tag{54}$$

We note that when $\tau_0^{(2)} = \tau_0^{(1)}$, the expression above reduces to the von Mises yield criterion.

Plots of the yield surfaces in the $(\bar{\tau}_n, \bar{\tau})$-space of applied stresses are given in Figs 10 and 11. Figure 10 shows the exact yield surface Φ for the choice of parameters, $\tau_0^{(2)}/\tau_0^{(1)} = 2$ and $c^{(2)} = 0.5$. The isotropic Reuss and Voigt (also known as Bishop–Hill estimate) bounds for the yield surfaces are also given for comparison. We note that the exact yield surface Φ is close to the Voigt upper bounding surface Φ_V for low values of the longitudinal shear stress ($\bar{\tau}_n < \frac{1}{2}\tau_0^{(1)}$), and close to the Reuss lower bounding surface Φ_R for

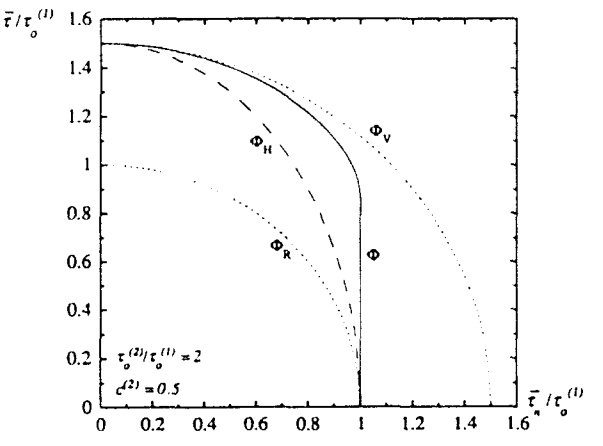

Fig. 10. Plots of the exact estimate, the anisotropic elliptic estimate of Hill, and the Voigt and Reuss isotropic estimates for the effective yield surface of a laminated composite with $\tau_0^{(2)}/\tau_0^{(1)} = 2$ and $c^{(2)} = 0.5$.

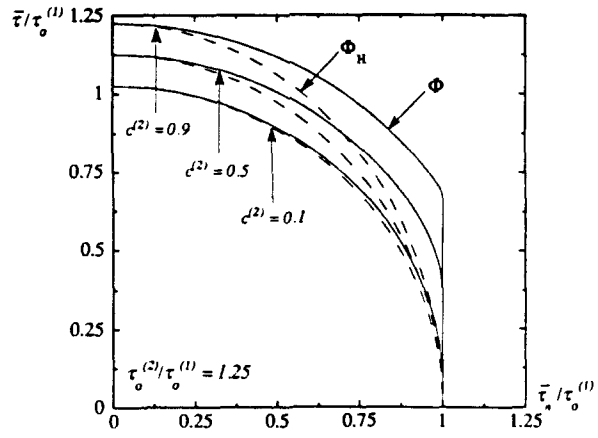

Fig. 11. Plots of the exact estimates (continuous lines) and the corresponding anisotropic elliptic estimates of Hill (dashed lines) for the effective yield surfaces of laminated composites with $\tau_0^{(2)}/\tau_0^{(1)} = 1.25$ and three different values of $c^{(2)}$ (0.1, 0.5 and 0.9).

low values of $\bar{\tau}$. We also include in Fig. 10 an estimate for the yield surface, Φ_H, which is based on the approximation of Hill (1951) for slightly anisotropic materials. This approximate yield surface is given by

$$\Phi_H(\bar{\sigma}) \equiv \frac{\bar{\tau}^2}{(c^{(1)}\tau_0^{(1)} + c^{(2)}\tau_0^{(2)})^2} + \frac{\bar{\tau}_n^2}{(\tau_0^{(1)})^2} - 1, \tag{55}$$

and we note that it amounts to an elliptic interpolation between the Voigt and Reuss yield functions.

We observe that for the largely anisotropic case depicted in Fig. 10 ($\tau_0^{(2)}/\tau_0^{(1)} = 2$ and $c^{(2)} = 0.5$), Hill's elliptic approximation severely underestimates the ultimate yield strength of the laminated composite for combined longitudinal and transverse loading. Figure 11 shows plots of the exact yield surfaces (continuous lines) and Hill's approximate yield surfaces (dashed lines) for a laminated composite with slight anisotropy ($\tau_0^{(2)}/\tau_0^{(1)} = 1.25$) and three values of $c^{(2)}$ (0.1, 0.5 and 0.9). For all values of $c^{(2)}$, the exact yield criterion bounds a larger region of the $(\bar{\tau}_n, \bar{\tau})$-plane than the Hill approximate criterion, and the two curves are only in good agreement for small volume fractions of the stronger phase ($c^{(2)} = 0.1$).

7. CLOSURE

In this paper, we have described the application of a new variational method, developed by Ponte Castañeda (1991a, 1992), to determine the effective constitutive behavior of laminated composites with elastoplastic phases in prescribed volume fractions. It constitutes one of the first applications of the method to composite materials with anisotropic symmetries [see also Ponte Castañeda (1992) and deBotton and Ponte Castañeda (1992) for the corresponding results for fiber-reinforced materials]. Because of the simplicity of the laminated microstructure, allowing for the determination of the *exact* effective properties of laminated composites, this work is of interest—not only on account of its practical significance—but also because it provides a simple case to evaluate the power of the new method. Additionally, the results of this paper suggest that when dealing with strongly anisotropic materials, it is *not* enough to consider the behavior of the composite under special loading conditions, since the behavior of the composite under different types of loading conditions may be *dramatically* different. Thus, we found that nonlinearity highlighted the differences in the constitutive response of laminated composites under transverse and longitudinal shear loading. Further, this study also underlined the significant coupling that may arise between different loading modes in nonlinear anisotropic composites. Thus, it was found that a small fixed preload of a laminate in the longitudinal direction leads to

continued increase of the longitudinal shear strain as the level of transverse shear stress (for example) is increased. It is anticipated that the features uncovered by the present analysis of nonlinear laminated composites will also be important in other types of nonlinear composites with anisotropic symmetries, such as the practically important class of fiber-reinforced materials.

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APPENDIX A: ON THE CHARACTERIZATION OF TRANSVERSELY ISOTROPIC MATERIALS

The purpose of this appendix is to gather some results relevant to the analysis of linearly elastic materials with transversely isotropic symmetry. These results are used extensively throughout the body of the paper in the development of effective stress/strain relations for nonlinear laminated composites. The emphasis of this section is on representations for the transversely isoropic invariants of the stress and strain tensors. The reason is that *nonlinear* transversely isotropic materials are most efficiently characterized in terms of energy-density functions depending on these invariants.

A.1. Isotropic invariants

It is well known that there are three isotropic invariants for a symmetric, second-order tensor. However, only two of these—those that are of quadratic order, or less—are relevant to linearly elastic behavior. These invariants may be expressed [see, for example, Walpole (1981)] in terms of two fourth-order *projection* tensors $\mathbf{J}$ and $\mathbf{K}$, such that $\mathbf{I} = \mathbf{J} + \mathbf{K}$, $\mathbf{JJ} = \mathbf{J}$, $\mathbf{KK} = \mathbf{K}$ and $\mathbf{JK} = \mathbf{0}$. Their Cartesian components are given by

$$J_{ijkl} = \tfrac{1}{3}\delta_{ij}\delta_{kl}, \quad K_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}), \tag{A1}$$

where δ_{ij} is the Kronecker delta symbol. Then, in terms of these projection tensors, we define two isotropic invariants of the stress tensor via

$$\sigma_m = \tfrac{1}{3}J_{kkij}\sigma_{ij} \quad \text{and} \quad \tau_e^2 = \tfrac{1}{2}K_{ijkl}\sigma_{ij}\sigma_{kl}, \tag{A2}$$

called the hydrostatic (mean) stress, and the effective shear stress, respectively. We also define the hydrostatic strain ε_m, and the effective shear strain γ_e by relations completely analogous to (A2).

It is important to note that the elasticity tensor $\mathbf{L}$ of an isotropic, linearly elastic material admits a spectral decomposition

$$\mathbf{L} = 3\kappa\mathbf{J} + 2\mu\mathbf{K}, \tag{A3}$$

where $\mathbf{J}$ and $\mathbf{K}$ play the role of the eigenprojections, and the bulk and shear moduli of the material, κ and μ, are the corresponding eigenvalues. As we will see next, the situation for transversely isotropic materials is different.

A.2. Transversely isotropic invariants

There are in general five transversely isotropic invariants of a symmetric, second-order tensor (Spencer, 1971). However, only four of these invariants are linear, or quadratic, in order. They may be represented in terms of the four projections tensors [see Walpole (1981)] $\mathbf{E}^{[1]}$, $\mathbf{E}^{[2]}$, $\mathbf{E}^{[3]}$ and $\mathbf{E}^{[4]}$, satisfying the relations $\mathbf{E}^{[p]}\mathbf{E}^{[p]} = \mathbf{E}^{[p]}$; $\mathbf{E}^{[p]}\mathbf{E}^{[q]} = \mathbf{0}$, $p \neq q$; and $\mathbf{E}^{[1]} + \mathbf{E}^{[2]} + \mathbf{E}^{[3]} + \mathbf{E}^{[4]} = \mathbf{I}$. The components of these four projections tensors are given respectively by

$$\begin{aligned}
E^{[1]}_{ijkl} &= \tfrac{1}{2}\beta_{ij}\beta_{kl}, \\
E^{[2]}_{ijkl} &= \alpha_{ij}\alpha_{kl}, \\
E^{[3]}_{ijkl} &= \tfrac{1}{2}(\beta_{ik}\beta_{jl} + \beta_{jk}\beta_{il} - \beta_{ij}\beta_{kl}), \\
E^{[4]}_{ijkl} &= \tfrac{1}{2}(\beta_{ik}\alpha_{jl} + \beta_{il}\alpha_{jk} + \beta_{jl}\alpha_{ik} + \beta_{jk}\alpha_{il}),
\end{aligned} \tag{A4}$$

where $\alpha_{ij} = n_i n_j$ and $\beta_{ij} = \delta_{ij} - n_i n_j$, with $\mathbf{n}$ denoting the axis of transverse isotropy. Then, the four transversely isotropic invariants of the stress tensor $\boldsymbol{\sigma}$ may be expressed in the forms

$$\begin{aligned}
\sigma_p &= \tfrac{1}{2}E^{[1]}_{ijkl}\sigma_{kl} = \tfrac{1}{2}\sigma_{ij}\beta_{ij}, && \{\tfrac{1}{2}(\sigma_{11} + \sigma_{22})\}, \\
\sigma_n &= E^{[2]}_{ijkl}\sigma_{kl} = \sigma_{ij}\alpha_{ij}, && \{\sigma_{33}\}, \\
\tau_p^2 &= \tfrac{1}{2}\sigma_{ij}E^{[3]}_{ijkl}\sigma_{kl} = \tfrac{1}{2}[\sigma_{ij}\sigma_{kl}\beta_{ik}\beta_{jl} - \tfrac{1}{2}(\sigma_{ij}\beta_{ij})^2], && \{\sigma_{12}^2 + \tfrac{1}{4}(\sigma_{11} - \sigma_{22})^2\}, \\
\tau_n^2 &= \tfrac{1}{2}\sigma_{ij}E^{[4]}_{ijkl}\sigma_{kl} = [\sigma_{ij}\sigma_{ki}\alpha_{jk} - (\sigma_{ij}\alpha_{ij})^2], && \{(\sigma_{13}^2 + \sigma_{23}^2)\},
\end{aligned} \tag{A5}$$

which correspond physically to the in-plane hydrostatic stress, the normal tensile stress, the (in-plane) transverse shear stress, and the (anti-plane) longitudinal shear stress (given in brackets are the corresponding representations for a choice of $\mathbf{n}$ aligned with the 3-direction). Analogous relations apply for the transversely isotropic invariants of the strain tensor $\boldsymbol{\varepsilon}$, denoted respectively ε_p, ε_n, γ_p and γ_n. We also note for latter reference that the following two relations hold between the transversely isotropic invariants of (A5) and the isotropic invariants of (A2), namely,

$$\sigma_m = \tfrac{1}{3}(2\sigma_p + \sigma_n), \quad \tau_e^2 = \tau_p^2 + \tau_n^2 + \tfrac{1}{3}(\sigma_p - \sigma_n)^2. \tag{A6}$$

Contrary to the situation for isotropic materials, the above four projection tensors are not the eigentensors of the spectral decomposition of an arbitrary transversely isotropic material (Mehrabadi and Cowin, 1990). Such eigentensors would unfortunately involve the material moduli. Therefore, it is necessary to introduce [see Walpole (1981)] two other tensors, that are *not* projections, $\mathbf{E}^{[5]}$ and $\mathbf{E}^{[6]}$, with components

$$E^{[5]}_{ijkl} = \alpha_{ij}\beta_{kl}, \qquad E^{[6]}_{ijkl} = \beta_{ij}\alpha_{kl}. \tag{A7}$$

Then, the elasticity tensor $\mathbf{L}$ of an arbitrary transversely isotropic material may be expressed in terms of these six tensors. It is worth mentioning that the above tensors satisfy the relation

$$\mathbf{J} = \tfrac{2}{3}\mathbf{E}^{[1]} + \tfrac{1}{3}\mathbf{E}^{[2]} + \tfrac{1}{3}(\mathbf{E}^{[5]} + \mathbf{E}^{[6]}), \tag{A8}$$

and that we can additionally define for later reference an additional tensor $\mathbf{E}'$ such that

$$\mathbf{E}' = \mathbf{E}^{[3]} + \mathbf{E}^{[4]} - \mathbf{K}. \tag{A9}$$

This last tensor is a projection tensor, which is orthogonal to $\mathbf{E}^{[3]}$ and $\mathbf{E}^{[4]}$.

Finally, we remark that the energy density function of a transversely isotropic, linearly elastic material may be represented in the form

$$U(\boldsymbol{\sigma}) = \psi(\sigma_p, \sigma_n, \tau_p, \tau_n), \tag{A10}$$

where ψ is a quadratic function. Then, the relation between the transversely isotropic stress and strain invariants is given by

$$\varepsilon_p = \frac{1}{2}\frac{\partial \psi}{\partial \sigma_p}, \quad \varepsilon_n = \frac{\partial \psi}{\partial \sigma_n}, \quad \gamma_p = \frac{1}{2}\frac{\partial \psi}{\partial \tau_p} \quad \text{and} \quad \gamma_n = \frac{1}{2}\frac{\partial \psi}{\partial \tau_n}. \tag{A11}$$

A.3. Incompressible, transversely isotropic invariants

For incompressible, transversely isotropic materials, it suffices to consider the three invariants of order less than quadratic on the space of traceless, symmetric, second-order tensors. These may be obtained in terms of the three orthogonal projection tensors $\mathbf{E}^{[3]}$, $\mathbf{E}^{[4]}$ and $\mathbf{E}'$, defined in the previous subsection. Thus, the incompressible, transversely isotropic invariants of the stress tensors $\boldsymbol{\sigma}$ are τ_p, τ_n, and the deviatoric shear stress

$$\tau_d = \frac{1}{\sqrt{3}}(\sigma_p - \sigma_n), \tag{A12}$$

corresponding to the three above projections, respectively. We note further that from (A6)$_2$ we have the following identity relating the effective shear stress and the incompressible, transversely isotropic invariants, $\tau_e^2 = \tau_p^2 + \tau_n^2 + \tau_d^2$. The corresponding strain invariants are denoted by γ_p, γ_n and γ_d.

Finally, we note that the elasticity tensor $\mathbf{L}$ of an incompressible, transversely isotropic, linearly elastic material admits a spectral decomposition of the form

$$\mathbf{L} = 2\mu_p\mathbf{E}^{[3]} + 2\mu_n\mathbf{E}^{[4]} + 2\mu_d\mathbf{E}', \tag{A13}$$

where μ_p, μ_n, μ_d, are the three shear moduli that suffice to characterize the behavior of such a material [see Lipton (1991a)].

APPENDIX B : A USEFUL IDENTITY

In this appendix, we demonstrate the following identity, which is used repeatedly in the body of the paper, namely

$$\frac{1}{\bar{\alpha}} = \min_{\omega^{(r)}, \bar{\omega}=1} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} (\omega^{(r)})^2 \right\}, \tag{B1}$$

where the variables $\alpha^{(r)} > 0$ $(r = 1, \ldots, n)$ are constant, and where the variables $\omega^{(r)}$ $(r = 1, \ldots, n)$ are subject to the constraint $\bar{\omega} = 1$.

We begin by letting g be the function defined by

$$g(\omega^{(r)}) = \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} (\omega^{(r)})^2. \tag{B2}$$

The choice of the set, $\omega^{(r)} = \alpha^{(r)}/\bar{\alpha}$, satisfies the constraint and is such that $g(\omega^{(r)}) = 1/\bar{\alpha}$. Consider next a second, arbitrary set, distinct from the first set, $\hat{\omega}^{(r)}$ $(r = 1, \ldots, n)$, such that $\bar{\hat{\omega}} = 1$, and let $\theta^{(r)} = \hat{\omega}^{(r)} - \omega^{(r)}$. Then, substitution of this second set into (B2) leads to

$$g(\hat{\omega}^{(r)}) = \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} (\hat{\omega}^{(r)})^2 = \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} (\omega^{(r)})^2 + \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} (\theta^{(r)})^2 > g(\omega^{(r)}), \tag{B3}$$

where we have used the fact that $\bar{\theta} = 0$. Hence, identity (B1) is demonstrated. In the body of the paper, we replace $\omega^{(r)}$ by $(1 - \omega^{(r)})$, with an appropriate modification for the constraint.

APPENDIX C: A SIMPLIFIED EXPRESSION FOR THE EFFCTIVE STRESS/STRAIN RELATIONS

Consider the following form for the effective energy function (26) of the incompressible laminated composite

$$\bar{U}(\bar{\sigma}) = \min_{\omega^{(r)},\bar{\omega}=0} \left\{ \sum_{r=1}^{n} c^{(r)}\psi^{(r)}(\tau^{(r)}) \right\}. \tag{C1}$$

where

$$\tau^{(r)} = \sqrt{(\bar{\tau}_p^2+\bar{\tau}_d^2)(1-\omega^{(r)})^2+\bar{\tau}_n^2}.$$

As shown in the body of the paper, we can eliminate the constraint $\bar{\omega}=0$ by letting

$$\omega^{(n)} = -\frac{1}{c^{(n)}}\sum_{r=1}^{n-1} c^{(r)}\omega^{(r)}, \tag{C2}$$

and rewriting (C1) in terms of the $n-1$ optimization variables $\omega^{(r)}$ $(r=1,\ldots,n-1)$ via

$$\bar{U}(\bar{\sigma}) = \min_{\substack{\omega^{(r)}\\ r=1,\ldots,n-1}} \left\{ \sum_{r=1}^{n-1} c^{(r)}\psi^{(r)}(\tau^{(r)}) + c^{(n)}\psi^{(n)}(\tau^{(n)}) \right\}, \tag{C3}$$

where the variables $\tau^{(s)}$ $(s=1,\ldots,n-1)$ are the same as before, but on the other hand

$$\tau^{(n)} = \sqrt{(\bar{\tau}_p^2+\bar{\tau}_d^2)\left(1+\frac{1}{c^{(n)}}\sum_{r=1}^{n-1} c^{(r)}\omega^{(r)}\right)^2+\bar{\tau}_n^2}.$$

Then, the $n-1$ optimization conditions of (C3) are given by the relations

$$-\frac{1}{\tau^{(r)}}(\psi^{(r)})'(\tau^{(r)})(1-\omega^{(r)})+\frac{1}{\tau^{(n)}}(\psi^{(n)})'(\tau^{(n)})\left(1+\frac{1}{c^{(n)}}\sum_{r=1}^{n-1} c^{(r)}\omega^{(r)}\right)=0, \quad (r=1,\ldots,n-1). \tag{C4}$$

If we now denote the optimal variables $\omega^{(r)}$, satisfying (C4), by $\hat{\omega}^{(r)}$ $(r=1,\ldots,n-1)$, the effective energy function of the incompressible laminate may then be written in the form

$$\bar{U}(\bar{\sigma}) = \sum_{r=1}^{n-1} c^{(r)}\psi^{(r)}(\hat{\tau}^{(r)}) + c^{(n)}\psi^{(n)}(\hat{\tau}^{(n)}), \tag{C5}$$

where

$$\hat{\tau}^{(s)} = \tau^{(s)}(\omega^{(s)}=\hat{\omega}^{(s)}) \quad \text{and} \quad \hat{\tau}^{(n)} = \sqrt{(\bar{\tau}_p^2+\bar{\tau}_d^2)\left(1+\frac{1}{c^{(n)}}\sum_{r=1}^{n-1} c^{(r)}\hat{\omega}^{(r)}\right)^2+\bar{\tau}_n^2}.$$

It follows that effective stress/strain relations of the laminated composite may be computed from the relations

$$\begin{aligned}\bar{\varepsilon} = &\sum_{r=1}^{n-1}\frac{c^{(r)}}{\hat{\tau}^{(r)}}(\psi^{(r)})'(\hat{\tau}^{(r)})\left[(1-\hat{\omega}^{(r)})^2\left(\bar{\tau}_p\frac{\partial\bar{\tau}_p}{\partial\bar{\sigma}}+\bar{\tau}_d\frac{\partial\bar{\tau}_d}{\partial\bar{\sigma}}\right)+\bar{\tau}_n\frac{\partial\bar{\tau}_n}{\partial\bar{\sigma}}\right]+\cdots\\ &\cdots+\frac{c^{(n)}}{\hat{\tau}^{(n)}}(\psi^{(n)})'(\hat{\tau}^{(n)})\left[\left(1+\frac{1}{c^{(n)}}\sum_{r=1}^{n-1} c^{(r)}\hat{\omega}^{(r)}\right)^2\left(\bar{\tau}_p\frac{\partial\bar{\tau}_p}{\partial\bar{\sigma}}+\bar{\tau}_d\frac{\partial\bar{\tau}_d}{\partial\bar{\sigma}}\right)+\bar{\tau}_n\frac{\partial\bar{\tau}_n}{\partial\bar{\sigma}}\right]+\cdots\\ &\cdots+\sum_{r=1}^{n-1} c^{(r)}(\bar{\tau}_p^2+\bar{\tau}_d^2)\frac{\partial\hat{\omega}^{(r)}}{\partial\bar{\sigma}}\left[-\frac{1}{\hat{\tau}^{(r)}}(\psi^{(r)})'(\hat{\tau}^{(r)})(1-\hat{\omega}^{(r)})+\frac{1}{\hat{\tau}^{(n)}}(\psi^{(n)})'(\hat{\tau}^{(n)})\left(1+\frac{1}{c^{(n)}}\sum_{r=1}^{n-1} c^{(r)}\hat{\omega}^{(r)}\right)\right].\end{aligned} \tag{C6}$$

We note that each of the terms in the last summation of (C6) is identical to zero by virtue of the optimizations conditions (C4). Thus, in the computation of the effective stress/strain relations, we may regard the optimizations variables as constants as far as derivatives with respect to $\bar{\sigma}$ are concerned, to obtain the final result

$$\bar{\varepsilon} = \sum_{r=1}^{n}\frac{c^{(r)}}{\hat{\tau}^{(r)}}(\psi^{(r)})'(\hat{\tau}^{(r)})\left[(1-\hat{\omega}^{(r)})^2\left(\bar{\tau}_p\frac{\partial\bar{\tau}_p}{\partial\bar{\sigma}}+\bar{\tau}_d\frac{\partial\bar{\tau}_d}{\partial\bar{\sigma}}\right)+\bar{\tau}_n\frac{\partial\bar{\tau}_n}{\partial\bar{\sigma}}\right], \tag{C7}$$

where $\hat{\omega}^{(n)}$ is defined via the relation (C2) in terms of the other $\hat{\omega}^{(r)}$ $(r=1,\ldots,n-1)$.

It can be shown that an analogous result may be obtained for the nonlinear *compressible* composite with effective energy function $\bar{U}$ given by (37). In fact, we may write the effective stress/strain relations for the nonlinear compressible laminate in the form

$$\tilde{\boldsymbol{\varepsilon}} = \sum_{r=1}^{n} c^{(r)} \frac{\partial \tilde{\psi}^{(r)}}{\partial \tilde{\boldsymbol{\sigma}}}(\tilde{\sigma}_e^{(r)}, \tilde{\sigma}_m^{(r)}), \tag{C8}$$

where $\tilde{\sigma}_e^{(r)}$ and $\tilde{\sigma}_m^{(r)}$ are evaluated from (36) at the optimal values of $\omega_e^{(r)}$ and $\omega_m^{(r)}$, denoted by $\tilde{\omega}_e^{(r)}$ and $\tilde{\omega}_m^{(r)}$, respectively. Here, the derivatives with respect to the average stress $\tilde{\boldsymbol{\sigma}}$ are evaluated with $\tilde{\omega}_e^{(r)}$ and $\tilde{\omega}_m^{(r)}$ fixed.

APPENDIX D: AN ALTERNATIVE DERIVATION OF EXPRESSION (37)

Having obtained expressions (35) and (37) for the effective energy functions of the linear and nonlinear laminated composites $\tilde{U}_0$ and $\tilde{U}$, respectively, we note that the form of these expressions is reminiscent of the type of result that one would expect from direct utilization of the principle of minimum complementary energy (4). In this appendix, we briefly show that results (35) and (37) for $\tilde{U}_0$ and $\tilde{U}$, respectively, can indeed be alternatively obtained from the principle of minimum complementary energy. It is important to emphasize, however, that while the derivations given in the body of the paper result from straightforward computations, the present derivations based on the principle of minimum complementary energy rely more directly on the physics of the problem, and were *motivated* by the prior derivations. Additionally, the case of a laminated composite is a very special microstructure; in general, we do *not* expect that we will be able to use the approach of this appendix for nonlinear composites with more general anisotropic microstructures.

We begin with the derivation of the linear result (35). We have already mentioned that the stress field within the laminated composite is piecewise constant, i.e. of the form

$$\boldsymbol{\sigma} = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x} \cdot \mathbf{n}) \boldsymbol{\sigma}^{(r)},$$

where $\boldsymbol{\sigma}^{(r)}$ corresponds to the constant stress field in phase r. The problem then reduces to that of finding these unknown phase stresses $\boldsymbol{\sigma}^{(r)}$, together with the corresponding constant strain fields $\boldsymbol{\varepsilon}^{(r)}$ (related to the stresses by the phase constitutive relations), and satisfying the conditions of continuity of the traction stresses and tangential strains across the interfaces between the phases, as well as the averaging conditions stated in Section 2.

In this connection, the interior and exterior projection operators of Hill (1972, 1983) $\mathbf{F} = \mathbf{E}^{[1]} + \mathbf{E}^{[3]}$ and $\mathbf{E} = \mathbf{E}^{[2]} + \mathbf{E}^{[4]}$ (refer to Appendix A), respectively, turn out to be useful because they allow the decomposition of any symmetric, second-order tensor into its tangential (interior) and traction (exterior) components (with reference to a boundary with normal $\mathbf{n}$). Thus, the tangential components of the strain (which must be continuous across interfacial boundaries on the laminated composite) are given by $\mathbf{F}\boldsymbol{\varepsilon}$, and, correspondingly, the traction components of the stress (which must also be continuous across the interfacial boundaries) are given by $\mathbf{E}\boldsymbol{\sigma}$. Alternatively, we may state that $\mathbf{E}^{[2]}\boldsymbol{\sigma}$, $\mathbf{E}^{[4]}\boldsymbol{\sigma}$ and $\mathbf{E}^{[1]}\boldsymbol{\varepsilon}$, $\mathbf{E}^{[3]}\boldsymbol{\varepsilon}$ must also be continuous across such boundaries.

Next, we apply the above results to the laminated composite, for which the interfacial boundaries are all perpendicular to a fixed vector $\mathbf{n}$. Since the traction stresses must be continuous from phase to phase, we have that

$$\mathbf{E}^{[2]}\boldsymbol{\sigma}^{(r)} = \mathbf{E}^{[2]}\tilde{\boldsymbol{\sigma}}, \quad \text{and} \quad \mathbf{E}^{[4]}\boldsymbol{\sigma}^{(r)} = \mathbf{E}^{[4]}\tilde{\boldsymbol{\sigma}}, \tag{D1}$$

where we have additionally made use of the average stress condition given in Section 2. We continue by noting that for an isotropic material (as are all the phases in the our laminate), $\mathbf{E}^{[3]}\boldsymbol{\sigma}^{(r)} = 2\mu_0^{(r)}\mathbf{E}^{[3]}\boldsymbol{\varepsilon}^{(r)}$ within each linear phase, and therefore for an isotropic phase the $\mathbf{E}^{[3]}$ projection of the stress tensor must have the same direction in all phases. Thus, applying the averaging condition for the stresses, we arrive at

$$\mathbf{E}^{[3]}\boldsymbol{\sigma}^{(r)} = (1 - \omega_e^{(r)})\mathbf{E}^{[3]}\tilde{\boldsymbol{\sigma}}, \tag{D2}$$

where the variables $\omega_e^{(r)}$ must satisfy the condition that $\tilde{\omega}_e = 0$. Additionally, since the $\mathbf{E}^{[3]}$-projection is one-dimensional, it follows that the $\mathbf{E}^{[1]}$-projections of the stress tensor must also be parallel from phase to phase. Therefore, applying the averaging condition for the stresses, we have that

$$\mathbf{E}^{[1]}\boldsymbol{\sigma}^{(r)} = (1 - \omega_m^{(r)})\mathbf{E}^{[1]}\tilde{\boldsymbol{\sigma}}, \tag{D3}$$

where the variables $\omega_m^{(r)}$ must satisfy the condition that $\tilde{\omega}_m = 0$. We note, however, that if $\mathbf{E}^{[1]}\tilde{\boldsymbol{\sigma}} = 0$ (or, equivalently, if $\tilde{\sigma}_p = 0$), the above result does not hold, because in this case the corresponding projections of the stress in the phases need not vanish (only their average needs to vanish).

Applying the results of Appendix A [in particular, (A6)], we conclude that the isotropic invariants of the stress tensor within each phase $\tau_e^{(r)}$ and $\sigma_m^{(r)}$ (on which the energy-density functions of each isotropic phase depend) are precisely those given by relations (36). Therefore, it follows from the principle of minimum complementary energy—by minimizing over the set of admissible stresses (i.e. over the optimizing variables $\omega_e^{(r)}$ and $\omega_m^{(r)}$ subject to the constraints $\tilde{\omega}_e = 0$ and $\tilde{\omega}_m = 0$)—that the effective energy function $\tilde{U}_0$ of the linear composite is indeed given by expression (35).

For the nonlinear laminated composite, we observe that the same analysis given above would also work, leading to expression (37) for $\tilde{U}$. The only modification that is required in this analysis is that for a nonlinear isotropic phase (say phase r), the relation $\mathbf{E}^{[3]}\boldsymbol{\sigma}^{(r)} = 2\mu_0^{(r)}\mathbf{E}^{[3]}\boldsymbol{\varepsilon}^{(r)}$ would not hold, but it can be easily shown that for the nonlinear isotropic material of the type considered in this work, the conclusion (D2) would still hold, and hence the final form for $\tilde{U}$ would be the same as that for the linear laminated composite $\tilde{U}_0$.

We conclude this appendix by stating an alternative form of (35) and (37) that works even when $\tilde{\sigma}_p = 0$. This is accomplished by redefining the optimizing variables $\omega_m^{(r)}$ in terms of the new variables

$$\dot{\omega}_m^{(r)} = \frac{\bar{\sigma}_p \omega_m^{(r)} - \bar{\sigma}_n}{\bar{\sigma}_p - \bar{\sigma}_n}, \tag{D4}$$

where we now need to have $\bar{\sigma}_p - \bar{\sigma}_n \neq 0$. In terms of the new variables $\dot{\omega}_m^{(r)}$ (the variables $\omega_e^{(r)}$ do not change), relation (37) is expressed in the form

$$\bar{U}(\bar{\sigma}) = \min_{\substack{\omega_e^{(r)}, \dot{\omega}_m^{(r)} \\ \dot{\omega}_e = \dot{\omega}_m = 0}} \left\{ \sum_{r=1}^{n} c^{(r)} \psi^{(r)}(\bar{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)}) \right\}, \tag{D5}$$

where

$$\bar{\tau}_e^{(r)} = \sqrt{(1-\omega_e^{(r)})^2 \bar{\tau}_p^2 + \bar{\tau}_n^2 + \tfrac{1}{3}[(\bar{\sigma}_n - \bar{\sigma}_p)(1-\dot{\omega}_m^{(r)})]^2} \quad \text{and} \quad \hat{\sigma}_m^{(r)} = \bar{\sigma}_n + \tfrac{2}{3}(\bar{\sigma}_p - \bar{\sigma}_n)(1-\dot{\omega}_m^{(r)}). \tag{D6}$$

Note that when $\bar{\sigma}_p = \bar{\sigma}_n = 0$, we are guaranteed that $\sigma_p^{(r)} = \sigma_n^{(r)} = 0$ in each phase, and then both forms are equally valid.